Differential Chapman-Kolmogorov Equation [nln56]

Focus on particular solutions of the (integral) Chapman-Kolmogorov equation that satisfy three conditions:

$$
\begin{aligned}
&\text{(i)} \quad \lim_{\Delta t\to 0}\frac{1}{\Delta t}P(x,t+\Delta t|x_0,t) = W(x|x_0;t) > 0,\\
&\text{(ii)} \quad \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x-x_0|<\epsilon} dx(x-x_0)P(x,t+\Delta t|x_0,t) = A(x_0,t) + \mathrm{O}(\epsilon),\\
&\text{(iii)} \quad \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x-x_0|<\epsilon} dx(x-x_0)^2P(x,t+\Delta t|x_0,t) = B(x_0,t) + \mathrm{O}(\epsilon).
\end{aligned}
$$

Comments:

- Integrals such as in (ii) and (iii) but with higher moments vanish,
- $W(x|x_0;t) > 0$ describes jumps,
- $A(x_0,t)$ describes drift,
- $B(x_0,t)$ describes diffusion.

Under assumptions including the ones stated above the following *differential Chapman-Kolmogorov equation* can be derived from its integral counterpart [see e.g. Gardiner 1985]:

$$
\begin{aligned}
\frac{\partial}{\partial t}P(x,t|x_0,t_0) = &-\frac{\partial}{\partial x}\Big[A(x,t)P(x,t|x_0,t_0)\Big] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[B(x,t)P(x,t|x_0,t_0)\Big]\\
&+\int dx'\Big[W(x|x';t)P(x',t|x_0,t_0) - W(x'|x;t)P(x,t|x_0,t_0)\Big].
\end{aligned}
$$

Initial condition: $P(x,t_0|x_0,t_0) = \delta(x-x_0)$.

Special cases:

- Drift equation: first term only. [nex29]
- Fokker-Planck equation: first and second terms only.
- Master equation: third term only.
- Diffusion process has $W = 0$, $A = 0$, $B \neq 0$. [nex27]
- Cauchy process has $W \neq 0$, $A = 0$, $B = 0$. [nex98]