Fokker-Planck Equation [nln57]

Extraction through systematic approximation of a (specific) Fokker-Planck equation from the (unspecific) Chapman-Kolmogorov equation for a homogeneous Markov process with continuous sample path.

Chapman-Kolmogorov equation: $P(x|x_0;t+\tau) = \int dx' P(x|x';\tau)P(x'|x_0;t)$.

Introduce a function $R(x)$ that is differentiable and vanishes at the boundaries of the range of x.

$$\int dx R(x)\frac{\partial}{\partial t}P(x|x_0;t) = \lim_{\tau\to 0}\frac{1}{\tau}\int dx R(x)\Big[P(x|x_0;t+\tau) - P(x|x_0;t)\Big] \quad (1)$$

$$= \lim_{\tau\to 0}\frac{1}{\tau}\int dx R(x)\left[\int dx' P(x|x';\tau)P(x'|x_0;t) - P(x|x_0;t)\right] \quad (2)$$

$$= \lim_{\tau\to 0}\frac{1}{\tau}\int dx\, P(x|x_0;t)\Big[\int dx' R(x')P(x'|x;\tau) - R(x)\Big] \quad (3)$$

$$= \lim_{\tau\to 0}\frac{1}{\tau}\int dx\, P(x|x_0;t)\int dx' P(x'|x;\tau) \times\left[(x'-x)R'(x) + \frac{1}{2}(x'-x)^2R''(x) + \ldots\right] \quad (4)$$

$$= \lim_{\tau\to 0}\frac{1}{\tau}\int dx\, R(x)\left[-\frac{\partial}{\partial x}\int dx'(x'-x)P(x'|x;\tau)P(x|x_0;t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\int dx'(x'-x)^2P(x'|x;\tau)P(x|x_0;t) + \ldots\right]. \quad (5)$$

(1) Construct partial time derivative;

(2) use Chapman-Kolmogorov equation;

(3) switch variables x, x';

(4) expand function $R(x)$ at position x;

(5) integrate by parts.

Since the above equation must hold for any $R(x)$ with the attributes mentioned the Fokker-Planck equation follows.

$$\frac{\partial}{\partial t}P(x|x_0;t) = -\frac{\partial}{\partial x}A(x)P(x|x_0;t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}B(x)P(x|x_0;t)$$

with drift and diffusion coefficients

$$A(x) = \lim_{\tau\to 0}\frac{1}{\tau}\int dx'(x'-x)P(x'|x;\tau),$$

$$B(x) = \lim_{\tau\to 0}\frac{1}{\tau}\int dx'(x'-x)^2P(x'|x;\tau).$$