Predominantly Small Jumps [nln58]

Jump processes are most commonly described by a master equation,

$$\frac{\partial}{\partial t} P(x,t|x_0) = \int dx' [W(x|x')P(x',t|x_0) - W(x'|x)P(x,t|x_0)].$$

If the transition rates favor small jumps such that their expansion in powers of jump size captures the essence of the process at hand we can extract a Fokker-Planck equation from the master equation.

Express transition rates as functions of jump size $\xi \doteq x' - x$:

$$W(x'|x) = \bar{W}(x;\xi), \quad W(x|x') = \bar{W}(x';-\xi).$$

Rewrite master equation:

$$\frac{\partial}{\partial t} P(x,t|x_0) = \int d\xi \Big[\bar{W}(x+\xi;-\xi)P(x+\xi,t|x_0) - \bar{W}(x;\xi)P(x,t|x_0)\Big].$$

Expand first term to second order:

$$\bar{W}(x;-\xi)P(x,t|x_0) + \xi\frac{\partial}{\partial x}\big[\bar{W}(x;-\xi)P(x,t|x_0)\big] + \frac{1}{2}\xi^2\frac{\partial^2}{\partial x^2}\big[\bar{W}(x;-\xi)P(x,t|x_0)\big].$$

Introduce jump moments:

$$\alpha_m(x) \doteq \int d\xi\, \xi^m \bar{W}(x;\xi) = \int dx' (x'-x)^m W(x'|x).$$

Substitution of expansion into master equation yields Fokker-Planck equation:

$$\frac{\partial}{\partial t} P(x,t|x_0) = -\frac{\partial}{\partial x}\big[\alpha_1(x)P(x,t|x_0)\big] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\big[\alpha_2(x)P(x,t|x_0)\big].$$

Comments:

- Convergent jump moments necessitate predominance of small jumps.
- The jump moments $\alpha_1(x)$ and $\alpha_2(x)$ only capture partial information contained in the transition rates $W(x|x')$, namely information associated with effective drift and effective diffusion.