Time Evolution of Mean and Variance [nln59]

Consider a stochastic process specified by the master equation

$$\frac{\partial}{\partial t} P(x,t|x_0) = \int dx'[W(x|x')P(x',t|x_0) - W(x'|x)P(x,t|x_0)].$$

Jump moments are extracted from the transition rates via

$$\alpha_m(x) \doteq \int dx'(x'-x)^m W(x'|x)$$

and assumed to be convergent at least for $m = 1, 2$ (see [nln58]).

Evaluate $\int dx x$[m.eq.] and $\int dx x^2$[m.eq.] to express the rate at which the first and second moments of x vary in time as follows:

$$\frac{d}{dt}\langle x\rangle = \int dx \int dx'(x'-x)W(x'|x)P(x,t|x_0),$$

$$\frac{d}{dt}\langle x^2\rangle = \int dx \int dx'(x'^2-x^2)W(x'|x)P(x,t|x_0).$$

Use $x'^2 - x^2 = (x'-x)^2 + 2x(x'-x)$ and the definition of jump moments to derive the following equations of motion:

$$\frac{d}{dt}\langle x\rangle = \langle\alpha_1(x)\rangle,$$

$$\frac{d}{dt}\langle x^2\rangle = \langle\alpha_2(x)\rangle + 2\langle x\alpha_1(x)\rangle.$$

Comments:

- If the jump moments are known and expandable in powers of x the expectation values on the right-hand sides become functions of $\langle x\rangle, \langle x^2\rangle, \ldots$
- In general, this leads to an infinite hierarchy of equations of motion for all moments $\langle x^m\rangle, m = 1, 2, \ldots$.
- In special cases, the ODEs for $\langle x\rangle$ and $\langle x^2\rangle$ form a closed set. Then they can be solved with no further approximations.
- The same equations of motions hold if the first two jump moments are replaced by the drift and diffusion coefficents of a Fokker-Planck equation, $A(x)$ and $B(x)$, respectively (see nln58]).
- Solvable cases are worked out in [nex30], [nex32].