Ornstein-Uhlenbeck Process [nln62]

The Fokker-Planck equation of the Ornstein-Uhlenbeck process,

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}(\kappa x P) + \frac{\gamma}{2}\frac{\partial^2 P}{\partial x^2}, \tag{1}$$

features the standard (constant) diffusion term, $B(x,t) = \gamma > 0$ and a position-dependent drift term, $A(x,t) = -\kappa x$ with $\kappa > 0$. This sort of drift is directed toward a particular position, namely $x = 0$. In a sense the drift counteracts the diffusion here. The diffusion term alone would broaden and flatten the probability distribution. A normal drift term, as in [nex101], has no effect on the broadening.

If a stationary solution $P_S(x)$ of (1) exists it must satisfy the equation

$$\frac{d}{dx}\left[\kappa x P_S(x) + \frac{\gamma}{2}\frac{d}{dx}P_S(x)\right] = 0. \tag{2}$$

Normalizability requires that $P_S(\pm\infty) = 0$ and $P'_S(\pm\infty) = 0$.
Consequence: the content of the square baracket in (2) must vanish. Separation of variables and integration then yields a Gaussian centered at $x = 0$,

$$P_S(x) = \sqrt{\frac{\kappa}{\pi\gamma}}\, e^{-\kappa x^2/\gamma}. \tag{3}$$

The ratio κ/γ in the prefactor and in the exponent represents the competition between diffusion and (restoring) drift.

For the dynamic solution of (1) we take two different approaches:

- [nex31] We derive from the second-order PDE (1) for the probability distribution $P(x,t)$ with initial condition $P(x,0) = \delta(x - x_0)$ a first-order PDE for the characteristic function and then solve that PDE. The result is a Gaussian whose mean and variance relax toward the values of the stationary solution (3). The relaxation rate is governed by the drift coefficient κ alone.

- [nex41] We search for solutions that permit a product ansatz $P(x,t) \doteq U(t)V(x)$ and aim to express the general solution as sum of such solutions. This is a reasonable goal for a linear PDE such as (1). The result expresses $U(t)$ as an exponential function and $V(x)$ as a Hermite polynomial multiplied by a Gaussian.