Einstein's Theory [nln65]

Theory operates on time scale dt, where $\Delta\tau_R \ll dt \ll \Delta\tau_{\rm O}$.
Focus on one space coordinate: x.
Local number density of Brownian particles: $n(x,t)$.
Brownian particles experience shift of size s in time dt.
Probability distribution of shifts: $P(s)$.
Successive shifts are assumed to be statistically independent.
Assumption justified by choice of time scale: $\Delta\tau_R \ll dt$.

Effect of shifts on profile of number density:

$$n(x,t+dt) = \int_{-\infty}^{+\infty} ds\, P(s) n(x+s,t).$$

Expansion of $n(x,t)$ in space and in time:

$$n(x+s,t) = n(x,t) + s\frac{\partial}{\partial x} n(x,t) + \frac{1}{2}s^2\frac{\partial^2}{\partial x^2} n(x,t) + \cdots\,,$$

$$n(x,t+dt) = n(x,t) + dt\frac{\partial}{\partial t} n(x,t) + \cdots$$

Integrals (normalization, reflection symmetry, diffusion coefficient):

$$\int_{-\infty}^{+\infty} ds\, P(s) = 1, \quad \int_{-\infty}^{+\infty} ds\, sP(s) = 0, \quad \frac{1}{2}\int_{-\infty}^{+\infty} ds\, s^2 P(s) \doteq Ddt.$$

Substitution of expansions with these integrals yields diffusion equation:

$$\frac{\partial}{\partial t} n(x,t) = D\frac{\partial^2}{\partial x^2} n(x,t).$$

Solution with initial condition $n(x,0) = N\delta(x-x_0)$ and no boundaries:

$$n(x,t) = \frac{N}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-x_0)^2}{4Dt}\right),$$

No drift: $\langle\langle x\rangle\rangle = 0$.

Diffusive mean-square displacement: $\langle\langle x^2\rangle\rangle = 2Dt$.