Einstein’s Theory

Theory operates on time scale $dt$, where $\Delta \tau_R \ll dt \ll \Delta \tau_O$.
Focus on one space coordinate: $x$.
Local number density of Brownian particles: $n(x, t)$.
Brownian particles experience shift of size $s$ in time $dt$.
Probability distribution of shifts: $P(s)$.
Successive shifts are assumed to be statistically independent.
Assumption justified by choice of time scale: $\Delta \tau_R \ll dt$.

Effect of shifts on profile of number density:

$$n(x, t + dt) = \int_{-\infty}^{+\infty} ds P(s)n(x + s, t).$$

Expansion of $n(x, t)$ in space and in time:

$$n(x + s, t) = n(x, t) + s \frac{\partial}{\partial x} n(x, t) + \frac{1}{2} s^2 \frac{\partial^2}{\partial x^2} n(x, t) + \cdots,$$

$$n(x, t + dt) = n(x, t) + dt \frac{\partial}{\partial t} n(x, t) + \cdots$$

Integrals (normalization, reflection symmetry, diffusion coefficient):

$$\int_{-\infty}^{+\infty} ds P(s) = 1, \quad \int_{-\infty}^{+\infty} ds s P(s) = 0, \quad \frac{1}{2} \int_{-\infty}^{+\infty} ds s^2 P(s) = D dt.$$

Substitution of expansions with these integrals yields diffusion equation:

$$\frac{\partial}{\partial t} n(x, t) = D \frac{\partial^2}{\partial x^2} n(x, t).$$

Solution with initial condition $n(x, 0) = N\delta(x - x_0)$ and no boundaries:

$$n(x, t) = \frac{N}{\sqrt{4 \pi D t}} \exp\left(-\frac{(x - x_0)^2}{4Dt}\right),$$

No drift: $\langle \langle x \rangle \rangle = 0$.

Diffusive mean-square displacement: $\langle \langle x^2 \rangle \rangle = 2Dt$. 