Smoluchowski Equation [nln66]

Einstein's result derived from different starting point.

Two laws relating number density and flux of Brownian particles:

(a) Conservation law: $\frac{\partial}{\partial t}n(x,t) = -\frac{\partial}{\partial x}j(x,t)$ (continuity equation);
local change in density due to net flux from or to vicinity.

(b) Constitutive law: $j(x,t) = -D\frac{\partial}{\partial x}n(x,t)$ (Fick's law);
flux driven by gradient in density.

Combination of (a) and (b) yields diffusion equation for density:

$$\frac{\partial}{\partial t}n(x,t) = D\frac{\partial^2}{\partial x^2}n(x,t). \tag{1}$$

Solution of (1) yields flux via (b).

Extension to include drift.

Brownian particles subject to external force $F_{\rm ext}(x,t)$.

Resulting drift velocity v, averaged over time scale dt identified in [nln65], produces drag force $F_{\rm drag} = -\gamma v$ due to front/rear asymmetry of collisions.

Damping constant: γ; mobility: γ^{-1}.

Drift contribution to flux $j(x,t)$ has general form $n(x,t)v(x,t)$.

On time scale dt of [nln65], forces are balanced: $F_{\rm ext} + F_{\rm drag} = 0$.

Drift velocity has reached terminal value: $v_{\rm T} = F_{\rm ext}/\gamma$.

(c) Extended constitutive law: $j(x,t) = -D\frac{\partial}{\partial x}n(x,t) + \gamma^{-1}F_{\rm ext}(x,t)n(x,t)$.

Substitution of (c) into (a) yields Smoluchowski equation:

$$\frac{\partial}{\partial t}n(x,t) = D\frac{\partial^2}{\partial x^2}n(x,t) - \gamma^{-1}\frac{\partial}{\partial x}\big[n(x,t)F_{\rm ext}(x,t)\big]. \tag{2}$$

The two terms on the rhs represent diffusion and drift, respectively.