Smoluchowski vs Fokker-Planck [nln68]

The Smoluchowski equation [nln66] as derived from a conservation law and constitutive law can be transcribed into a Fokker-Planck equation [nln57] if density and flux of particles are replaced by density and flux of probability.

Here we use generic notation:

- density: $\rho(x,t)$,
- flux: $J(x,t)$,
- diffusivity: $D(x)$,
- mobility: Γ,
- external force: $F(x)$.

Conservation law: $\dfrac{\partial}{\partial t}\rho(x,t) = -\dfrac{\partial}{\partial x}J(x,t)$.

Constitutive law: $J(x,t) = -D(x)\dfrac{\partial}{\partial x}\rho(x,t) + \Gamma F(x)\rho(x,t)$.

$$\Rightarrow \; \frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x}\Big[\Gamma F(x)\rho(x,t)\Big] + \underbrace{\frac{\partial}{\partial x}\left[D(x)\frac{\partial}{\partial x}\rho(x,t)\right]}.$$

$$\frac{\partial^2}{\partial x^2}\Big[D(x)\rho(x,t)\Big] = \frac{\partial}{\partial x}\Big[D'(x)\rho(x,t)\Big] + \overbrace{\frac{\partial}{\partial x}\left[D(x)\frac{\partial}{\partial x}\rho(x,t)\right]}.$$

$$\Rightarrow \; \frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x}\Big[\Big(\underbrace{\Gamma F(x) + D'(x)}_{A(x)}\Big)\rho(x,t)\Big] + \frac{\partial^2}{\partial x^2}\Big[\underbrace{D(x)}_{B(x)}\rho(x,t)\Big].$$

$A(x)$ and $B(x)$ represent drift and diffusion in the Fokker-Planck equation.