Multivariate Distributions [nln7]

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector variable with n components.

Joint probability distribution: $P(x_1, \ldots, x_n)$.

Marginal probability distribution:

$$P(x_1, \ldots, x_m) = \int dx_{m+1} \cdots dx_n \, P(x_1, \ldots, x_n).$$

Conditional probability distribution: $P(x_1, \ldots, x_m | x_{m+1}, \ldots, x_n)$.

$$P(x_1, \ldots, x_n) = P(x_1, \ldots, x_m | x_{m+1}, \ldots, x_n) P(x_{m+1}, \ldots, x_n).$$

Moments: $\langle X_1^{m_1} \cdots X_n^{m_n} \rangle = \int dx_1 \cdots dx_n \, x_1^{m_1} \cdots x_n^{m_n} P(x_1, \ldots, x_n)$.

Characteristic function: $\Phi(\mathbf{k}) = \langle e^{i\mathbf{k}\cdot\mathbf{X}} \rangle$.

Moment expansion: $\Phi(\mathbf{k}) = \displaystyle\sum_0^\infty \frac{(ik_1)^{m_1} \cdots (ik_n)^{m_n}}{m_1! \ldots m_n!} \langle X_1^{m_1} \cdots X_n^{m_n} \rangle$.

Cumulant expansion: $\ln \Phi(\mathbf{k}) = \displaystyle{\sum_0^\infty}' \frac{(ik_1)^{m_1} \cdots (ik_n)^{m_n}}{m_1! \ldots m_n!} \langle\langle X_1^{m_1} \cdots X_n^{m_n} \rangle\rangle$.
(prime indicates absence of term with $m_1 = \cdots = m_n = 0$).

Covariance matrix: $\langle\langle X_i X_j \rangle\rangle = \langle (X_i - \langle X_i \rangle)(X_j - \langle X_j \rangle) \rangle$.
($i = j$: variances, $i \neq j$: covariances).

Correlations: $C(X_i, X_j) = \dfrac{\langle\langle X_i X_j \rangle\rangle}{\sqrt{\langle\langle X_i \rangle\rangle \langle\langle X_j \rangle\rangle}}$.

Statistical independence of X_1, X_2: $P(x_1, x_2) = P_1(x_1) P_2(x_2)$.

Equivalent criteria for statistical independence:

- all moments factorize: $\langle X_1^{m_1} X_2^{m_2} \rangle = \langle X_1^{m_1} \rangle \langle X_2^{m_2} \rangle$;
- characteristic function factorizes: $\Phi(k_1, k_2) = \Phi_1(k_1) \Phi_2(k_2)$;
- all cumulants $\langle\langle X_1^{m_1} X_2^{m_2} \rangle\rangle$ with $m_1 m_2 \neq 0$ vanish.

If $\langle\langle X_1 X_2 \rangle\rangle = 0$ then X_1, X_2 are called *uncorrelated*.
This property does not imply *statistical independence*.