Langevin's Theory [nln71]

Langevin's theory of Brownian motion operates on a less contracted level of description than Einstein's theory [nln65]. The operational time scale is small compared to the relaxation time: $dt \ll \Delta\tau_{\rm R}$. [nln64]. On this time scale inertia matters, implying that velocity cannot change abruptly. Velocity and position variables are kinematically coupled.

The Langevin equation,

$$m\ddot{x} = -\gamma\dot{x} + f(t), \tag{1}$$

is constructed from Newton's second law with two forces acting:

- drag force: $-\gamma\dot{x}$ (parametrized by mobility γ^{-1}),
- random force: $f(t)$ (Gaussian white noise/Wiener process).

Since we do not know $f(t)$ explicitly we cannot solve (1) for $x(t)$. However, we know enough about $f(t)$ to solve (1) for $\langle x^2\rangle$ as a function of time [nex118].

First step: derive the linear, 2nd-order ODE for $\langle x^2\rangle$,

$$m\frac{d^2}{dt^2}\langle x^2\rangle + \gamma\frac{d}{dt}\langle x^2\rangle = 2k_BT, \tag{2}$$

using

- the white-noise implication that the random force and the position are uncorrelated, $\langle xf(t)\rangle$,
- the equilibrium implication that the average kinetic energy of the Brownian particle satisfies equipartition, $\langle\dot{x}^2\rangle = k_{\rm B}T/m$.

Second step: Integrate (2) twice using

- initial conditions $\langle x^2\rangle_0 = 0$ and $d\langle x^2\rangle_0/dt = 0$,
- Einstein's fluctuation-dissipation relation $D = k_{\rm B}T/\gamma$,
- the fact that (2) is a 1st-order ODE for $d\langle x^2\rangle/dt$.

The result reads

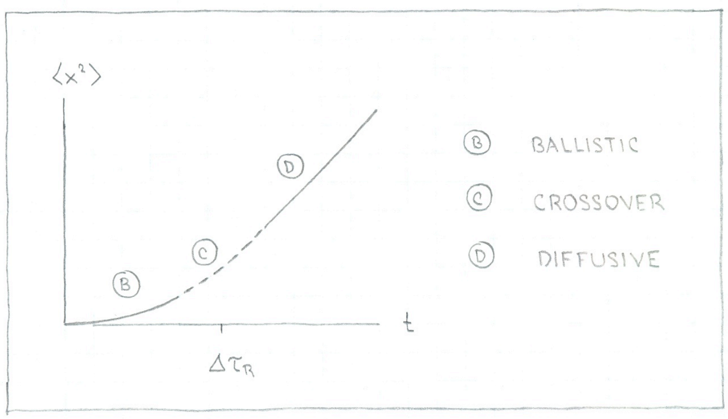
$$\langle x^2\rangle = 2D\left[t - \frac{m}{\gamma}\left(1 - e^{-\gamma t/m}\right)\right]. \tag{3}$$

Within the framework of Langevin's theory, the relaxation time previously identified [nln64] is

$$\Delta\tau_{\rm R} = \frac{m}{\gamma}.$$

This relaxation time separates short-time *ballistic* regime from a long-time *diffusive* regime:

- $t \ll \dfrac{m}{\gamma}: \quad \langle x^2\rangle \sim \dfrac{D\gamma}{m}t^2 = \dfrac{k_BT}{m}t^2 = \langle v^2\rangle t^2,$
- $t \gg \dfrac{m}{\gamma}: \quad \langle x^2\rangle \sim 2Dt.$



Applications and variations:

▷ Mean-square displacement of Brownian particle [nex56] [nex57] [nex118]

▷ Formal solution of Langevin equation [nex53]

▷ Velocity correlation function of Brownian particle [nex55] [nex119] [nex120]