Diffusion Equation Analyzed [nln73]

Here we present two simple and closely related methods of analyzing the diffusion equation,

$$\frac{\partial}{\partial t}\rho(x,t) = D\frac{\partial^2}{\partial x^2}\rho(x,t), \tag{1}$$

in one dimension and with no boundary constraints.

Fourier transform:

Ansatz for plane-wave solution: $\rho(x,t)_k = \tilde{\rho}_k(t)\, e^{ikx}$.

Substitution of ansatz into PDE (1) yields ODE for Fourier amplitude $\tilde{\rho}_k(t)$, which is readily solved:

$$\frac{d}{dt}\tilde{\rho}_k(t) = -Dk^2\tilde{\rho}_k(t) \quad \Rightarrow\ \tilde{\rho}_k(t) = \tilde{\rho}_k(0)\, e^{-Dk^2 t}.$$

Initial Fourier amplitudes from initial distribution:

$$\tilde{\rho}_k(0) = \int_{-\infty}^{+\infty} dx\, e^{-ikx}\rho(x,0). \tag{2}$$

Time-dependence of distribution as superposition of plane-wave solutions:

$$\rho(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ikx}\tilde{\rho}_k(0)\, e^{-Dk^2 t}. \tag{3}$$

Green's function:

Green's function $G(x,t)$ describes time evolution of point source at $x=0$:

$$G(x,0) = \delta(x) \quad \Rightarrow\ \tilde{G}_k(0) \doteq \int_{-\infty}^{+\infty} dx\, e^{-ikx}G(x,0) = 1 \quad \Rightarrow\ \tilde{G}_k(t) = e^{-Dk^2 t}.$$

$$\Rightarrow\ G(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ikx-Dk^2 t} = \frac{1}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}. \tag{4}$$

Superposition of point-source solutions in the form of a convolution integral:

$$\rho(x,t) = \int_{-\infty}^{+\infty} dx'\rho(x',0)G(x-x',t). \tag{5}$$