Brownian Harmonic Oscillator [nln75]

A Brownian particle of mass m, immersed in a fluid, is constrained to move along the x-axis and subject to a restoring force. The motion is simultaneously propelled and attenuated by the fluid particles. We analyze this system in several different ways with consistent results.

Two equivalent specifications of the system by Langevin-type equations:

- Attenuation without memory and white-noise:

$$m\ddot{x} + \gamma\dot{x} + kx = f_{\mathrm{w}}(t), \tag{1}$$

  γ: damping constant representing instantaneous attenuation,

  $k = m\omega_0^2$: stiffness of the restoring force,

  $f_{\mathrm{w}}(t)$: white-noise random force.

- Attenuation with memory and correlated-noise:

$$m\frac{dx}{dt} + \int_{-\infty}^{t} dt'\alpha(t-t')x(t') = \frac{1}{\omega_0}f_{\mathrm{c}}(t), \tag{2}$$

  $\alpha(t) = m\omega_0^2 e^{-(\gamma/m)t}$: attenuation function,

  $f_{\mathrm{c}}(t)$: correlated-noise random force.

The random forces and their associated types of attenuation must satisfy the the fluctuation-dissipation relation of [nln72].

Tasks carried out in a series of exercises:

- Equivalence of (1) and (2) shown via derivation of (1) from (2) [nex129].
- Fourier analysis of (1) yields spectral density $S_{xx}(\omega)$ for position variable of Brownian particle [nex121].
- Position correlation function $\langle x(t)x(0)\rangle$ via inverse Fourier transform. Cases of underdamping, critical damping, and overdamping [nex122].
- Calculation of $S_{xx}(\omega)$ from (2) and $\langle x(t)x(0)\rangle$ via contour integrals [nex123]. Physical significance of pole structure in $S_{xx}(\omega)$.
- Spectral density $S_{vv}(\omega)$ and correlation function $\langle v(t)v(0)\rangle$ for velocity variable of Brownian particle [nex58].
- Langevin-type equation for velocity variable $v(t)$ and formal solution of that equation [nex59].
- Nonequilibrium velocity correlation function $\langle v(t_2)v(t_1)\rangle$ and stationarity limit $t_1, t_2 \to \infty$ with $0 < t_2 - t_1 < \infty$ [nex60].