Binomial, Poisson, and Gaussian Distributions [nln8]

Consider a set of N independent experiments, each having two possible outcomes occurring with given probabilities.

events	$A + B = S$
probabilities	$p + q = 1$
random variables	$n + m = N$

Binomial distribution:

$$P_N(n) = \frac{N!}{n!(N-n)!}\, p^n (1-p)^{N-n}.$$

Mean value: $\langle n \rangle = Np$.

Variance: $\langle\langle n^2 \rangle\rangle = Npq$. [nex15]

In the following we consider two different asymptotic distributions in the limit $N \to \infty$.

Poisson distribution:

Limit #1: $N \to \infty$, $p \to 0$ such that $Np = \langle n \rangle = a$ stays finite [nex15].

$$P(n) = \frac{a^n}{n!}\, e^{-a}.$$

Cumulants: $\langle\langle n^m \rangle\rangle = a$.

Factorial cumulants: $\langle\langle n^m \rangle\rangle_f = a\delta_{m,1}$. [nex16]

Single parameter: $\langle n \rangle = \langle\langle n^2 \rangle\rangle = a$.

Gaussian distribution:

Limit #2: $N \gg 1$, $p > 0$ with $Np \gg \sqrt{Npq}$.

$$P_N(n) = \frac{1}{\sqrt{2\pi\langle\langle n^2 \rangle\rangle}} \exp\left(-\frac{(n - \langle n \rangle)^2}{2\langle\langle n^2 \rangle\rangle} \right).$$

Derivation: DeMoivre-Laplace limit theorem [nex21].

Two parameters: $\langle n \rangle = Np$, $\langle\langle n^2 \rangle\rangle = Npq$.

Special case of central limit theorem [nln9].