Moment Expansion vs Continued Fraction I [nln85]

Moment expansions of fluctuation function (see [nln78]):

$$C_0(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} M_{2k} t^{2k}, \quad M_0 = 1. \tag{1}$$

Asymptotic expansion and continued-fraction representation of relaxation function:

$$c_0(z) \doteq \int_0^{\infty} dt\, e^{-zt} C_0(t) = \sum_{k=0}^{\infty} M_{2k} z^{-(2k+1)} = \cfrac{1}{z + \cfrac{\Delta_1}{z + \cfrac{\Delta_2}{z + \cdots}}}. \tag{2}$$

Transformation relations between $\{M_{2k}\}$ and $\{\Delta_n\}$ extracted from inspecting the two representations of $c_0(z)$, using the algebraic equations in [nln84].

Forward direction: $\{M_{2k}\} \;\rightarrow\; \{\Delta_n\}$

Set values: $M_{2k}^{(0)} \doteq M_{2k}, \quad M_{2k}^{(-1)} \doteq 0, \quad k = 1, 2, \ldots, K; \quad \Delta_{-1} = \Delta_0 \doteq 1.$

Then evaluate

$$M_{2k}^{(n)} = \frac{M_{2k}^{(n-1)}}{\Delta_{n-1}} - \frac{M_{2k-2}^{(n-2)}}{\Delta_{n-2}}, \quad \Delta_n = M_{2n}^{(n)} \tag{3}$$

sequentially for $k = n, n+1, \ldots, K$ and $n = 1, 2, \ldots, K$.

Reverse direction: $\{\Delta_n\} \;\rightarrow\; \{M_{2k}\}$

Set values: $M_{2k}^{(k)} \doteq \Delta_k, \quad M_{2k}^{(-1)} \doteq 0, \quad k = 1, 2, \ldots, K; \quad \Delta_{-1} = \Delta_0 \doteq 1.$

Then evaluate

$$M_{2k}^{(n-1)} = \Delta_{n-1} M_{2k}^{(n)} + \frac{\Delta_{n-1}}{\Delta_{n-2}} M_{2k-2}^{(n-2)}, \quad M_{2k} = M_{2k}^{(0)} \tag{4}$$

sequentially for $n = k, k-1, \ldots, 1$ and $k = 1, 2, \ldots, K$.

Forward direction: $\{M_{2k}\} \;\rightarrow\; \{\Delta_n\}$

$k \backslash n$	0	1	2	3	4
0	1	M_2	M_4	M_6	M_8
1		M_2	M_4	M_6	M_8
2			$\frac{M_4}{M_2} - M_2$	$\frac{M_6}{M_2} - M_4$	$\frac{M_8}{M_2} - M_6$
3				$\dfrac{\frac{M_6}{M_2} - M_4}{\frac{M_4}{M_2} - M_2} - \frac{M_4}{M_2}$	$\dfrac{\frac{M_8}{M_2} - M_6}{\frac{M_4}{M_2} - M_2} - \frac{M_6}{M_2}$

Reverse direction: $\{\Delta_n\} \;\rightarrow\; \{M_{2k}\}$

$n \backslash k$	0	1	2	3
0	1	Δ_1	$\Delta_1(\Delta_1 + \Delta_2)$	$\Delta_1[\Delta_2\Delta_3 + (\Delta_1 + \Delta_2)^2]$
1		Δ_1	$\Delta_1(\Delta_1 + \Delta_2)$	$\Delta_1[\Delta_2\Delta_3 + (\Delta_1 + \Delta_2)^2]$
2			Δ_2	$\Delta_2(\Delta_1 + \Delta_2 + \Delta_3)$
3				Δ_3

First two steps in forward directions spelled out:

- $c_0(z) = z^{-1} - M_2 z^{-3} + M_4 z^{-5} - M_6 z^{-7} + \ldots$
- $\Delta_1 c_1(z) = 1 - z c_0(z) = \underbrace{M_2}_{\Delta_1} z^{-2} - M_4 z^{-4} + M_6 z^{-6} - \ldots$
- $c_1(z) = z^{-2} - \frac{M_4}{M_2} z^{-4} + \frac{M_6}{M_2} z^{-6} - \frac{M_8}{M_2} z^{-8} + \ldots$
- $\Delta_2 c_2(z) = c_0(z) - z c_1(z) = \underbrace{\left(\frac{M_4}{M_2} - M_2\right)}_{\Delta_2} z^{-3} - \left(\frac{M_6}{M_2} - M_4\right) z^{-5} + \ldots$