# Link to Green's Function Formalism [nln88]

Retarded Green's function (two varieties):[1][2][3]

$$\tilde{G}_\pm(t-t') \doteq \langle\langle A(t); B(t')\rangle\rangle^\pm = -\imath\theta(t-t')\langle[A(t), B(t')]_\pm\rangle. \tag{1}$$

Equations of motion are hierarchical in nature:

$$\imath\frac{\partial}{\partial t}\langle\langle A(t); B(t')\rangle\rangle^\pm = \delta(t-t')\langle[A,B]_\pm\rangle + \langle\langle [A(t),\mathcal{H}]_-; B(t')\rangle\rangle^\pm. \tag{2}$$

Frequency-dependent Green's functions via Fourier transform:

$$G_\pm(\omega) \doteq \langle\langle A; B\rangle\rangle^\pm_\omega = \int_{-\infty}^{+\infty} dt\, e^{\imath\omega t}\tilde{G}_\pm(t), \tag{3}$$

$$\omega\langle\langle A; B\rangle\rangle^\pm_\omega = \langle[A,B]_\pm\rangle + \langle\langle [A,\mathcal{H}]_-; B\rangle\rangle^\pm_\omega. \tag{4}$$

Spectral representations relate the Green's functions $G_\pm(\zeta)$ to the functions $S(\omega)$, $\Phi(\omega)$, and $\chi''(\omega)$ as defined in [nln39]:

$$G_\pm(\zeta) = \int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}\frac{S(\bar{\omega})}{\zeta-\bar{\omega}}\big[1 \pm e^{-\beta\bar{\omega}}\big], \qquad \zeta = \omega + \imath\epsilon, \quad \epsilon > 0, \tag{5a}$$

$$G_+(\zeta) = 2\int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}\frac{\Phi(\bar{\omega})}{\zeta-\bar{\omega}}, \quad G_-(\zeta) = 2\int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}\frac{\chi''(\bar{\omega})}{\zeta-\bar{\omega}}, \tag{5b}$$

Inverse relations:

- $\Phi(\omega) = -\lim_{\epsilon\to 0}\mathrm{Im}\big[G_+(\omega+\imath\epsilon)\big]$ (spectral density)
- $\chi''(\omega) = -\lim_{\epsilon\to 0}\mathrm{Im}\big[G_-(\omega+\imath\epsilon)\big]$ (dissipation function)
- $S(\omega) = \dfrac{\Phi(\omega)}{2(1+e^{-\beta\omega})} = \dfrac{\chi''(\omega)}{2(1-e^{-\beta\omega})}$ (structure function)

Relation to relaxation function for $\imath\zeta = -z$ (see [nln31] and [nln84]):

$$c_0(z) = \int_0^\infty dt\, e^{\imath\zeta t}\frac{\tilde{\Phi}(t)}{\tilde{\Phi}(0)} = \frac{\imath}{\tilde{\Phi}(0)}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Phi(\omega)}{\zeta-\omega} = \frac{\imath}{2\tilde{\Phi}(0)}G_+(\zeta).$$

[1] Here we set $\hbar = 1$ as is common practice.

[2] The bracket $[\ ,\ ]_-$ stands for commutators and the bracket $[\ ,\ ]_+$ for anticommutators.

[3] One variety coincides with Kubo's response function from [nln27]: $\tilde{G}_-(t) = -\tilde{\chi}_{AB}(t)$.