Structure Function [nln92]

Laplace transform (with $\imath\zeta = -z$):

$$d_k(\zeta) \doteq \int_0^\infty dt\, e^{\imath\zeta t} D_k(t). \tag{1}$$

Coupled ODEs for $D_k(t)$ become coupled algebraic equations for $d_k(\zeta)$:

$$(\zeta - a_k)d_k(\zeta) - \imath\delta_{k,0} = d_{k-1}(\zeta) + b_{k+1}^2 d_{k+1}(\zeta), \quad k = 0, 1, 2, \ldots \tag{2}$$

Condition: $d_{-1}(\zeta) \equiv 0$.

Recursive construction of continued fraction representation for $d_0(\zeta)$:

$$k = 0:\ (\zeta - a_0)d_0(\zeta) - \imath = b_1^2 d_1(\zeta) \quad \Rightarrow\ d_0(\zeta) = \frac{\imath}{\zeta - a_0 - b_1^2 \dfrac{d_1(\zeta)}{d_0(\zeta)}},$$

$$k = 1:\ (\zeta - a_1)d_1(\zeta) = d_0(\zeta) + b_2^2 d_2(\zeta) \quad \Rightarrow\ \frac{d_1(\zeta)}{d_0(\zeta)} = \frac{1}{\zeta - a_1 - b_2^2 \dfrac{d_2(\zeta)}{d_1(\zeta)}},$$

$$\Rightarrow\ d_0(\zeta) = \cfrac{\imath}{\zeta - a_0 - \cfrac{b_1^2}{\zeta - a_1 - \cfrac{b_2^2}{\zeta - a_2 - \cdots}}} \tag{3}$$

Structure function $S(\omega)$ from $d_0(\zeta)$: combine Fourier transform with inverse Laplace transform.

$$S_0(\omega) \doteq \int_{-\infty}^{+\infty} dt\, e^{\imath\omega t} D_0(t),$$

$$D_0(t) = -\frac{1}{2\pi}\int_{\mathcal{C}} d\zeta\, e^{-\imath\zeta t} d_0(\zeta)$$

$$\Rightarrow\ S_0(\omega) = 2\lim_{\epsilon\to 0} \mathrm{Re}[d_0(\omega + \imath\epsilon)]. \tag{4}$$