[nex100] Random walk in one dimension: tiny steps at frequent times

Consider the conditional probability distribution $P(n, t_N|0, 0)$ describing a biased random walk in one dimension as determined by the (discrete) Chapman-Kolmogorov equation,

$$P(n, t_{N+1}|0, 0) = \sum_m P(n, t_{N+1}|m, t_N)P(m, t_N|0, 0),$$

where

$$P(n, t_{N+1}|m, t_N) = p\delta_{m,n-1} + q\delta_{m,n+1}$$

expresses the instruction that the walker takes a step of fixed size forward (with probability p) or backward (with probability $q = 1 - p$) after one time unit. Set $n\ell = x_n$ for the position of the walker and $N\tau = t_N$ for the elapsed time, where ℓ is the (constant) step size and τ is the length of the (constant) time unit between steps. Now consider the limit $\ell \to 0$, $\tau \to 0$, $p - q \to 0$ such that $\ell^2/2\tau = D$ and $(p - q)\ell/\tau = v$. Show that this limit converts the Chapman-Kolmogorov equation into a Fokker-Planck equation with constant coefficients v (drift) and D (diffusion). What is the solution $P(x, t|0, 0)$ in this limit?

Solution: