In the chemical reaction \( A + X \leftrightarrow A + Y \), the molecule \( A \) is a catalyst at constant concentration. The total number of reacting molecules, \( n_x + n_y = N \), is also constant. \( K_1 \) is the probability per unit time that a molecule \( X \) interacts with a molecule \( A \) to turn into a molecule \( Y \), and \( K_2 \) is the probability per unit time that a \( Y \) interacts with an \( A \) to produce an \( X \). The dynamics may be described by a master equation for \( P(n,t) \), where \( n \equiv n_x, n_y = N - n \). The transition rates are \( W(m|n) = K_1 n \delta_{m,n-1} + K_2 (N-n) \delta_{m,n+1} \). The total rate of chemical reactions is defined as follows:

\[
R(t) = \sum_{nm} W(n|m)P(m,t).
\]

(a) Express \( R(t) \) in terms of \( \langle n(t) \rangle \).
(b) Use the result of \( \langle n(\infty) \rangle \) from [nex46] to calculate the total rate of chemical reactions in the stationary state. Set \( K_1 = \gamma, K_2 = 1 - \gamma \) and compare the \( \gamma \)-dependence of \( R(\infty) \) with that of \( \langle n^2(\infty) \rangle \) from [nex46], which is a measure of the fluctuations in the population of molecules.
(c) Use the result of \( \langle n(t) \rangle \) from [nex107] to calculate the time evolution of \( R(t) \). Plot \( R(t) \) for \( n_0 = 0, K_1 = \gamma, K_2 = 1 - \gamma \) and various fixed values of \( \gamma \). The time scale is thus set. Compare the graph of \( R(t) \) with the graph of \( \langle n^2(t) \rangle \) from [nex107]. Explain the similarities and differences.

Solution: