Consider the master equation of the birth-death process with transition rates,

\[ W(m|n) = n\lambda \delta_{m,n+1} + \left[ n\mu + \frac{\gamma}{N} n(n-1) \right] \delta_{m,n-1}. \]

It describes a population with a linear birth rate, \( n\lambda \), and a linear death rate, \( n\mu \), as in [nex44]. To account for the unhealthy environment in crowded conditions \( (n \approx N) \), a nonlinear death rate has been added. The nonlinearity suppresses the continued exponential growth found in [nex44] for the case \( \lambda > \mu \) and stabilizes a stationary state.

(a) Construct the equations of motion for \( \langle n(t) \rangle \) from the first jump moment as in [nex44]. Then neglect fluctuations by setting \( \langle n^2(t) \rangle \approx \langle n(t) \rangle^2 \) and \( \langle n(t) \rangle = Nx(t) + O(\sqrt{N}) \) to arrive (in leading order) at the Malthus-Verhulst equation,

\[ \frac{dx}{dt} = (\lambda - \mu)x - \gamma x^2, \]

for the time-dependence of the (scaled) average population. Derive the analytic solution,

\[ x(t) = \frac{x_0 e^{(\lambda - \mu)t}}{1 + \frac{\gamma x_0}{\lambda - \mu} \left[ e^{(\lambda - \mu)t} - 1 \right]}, \]

pertaining to initial value \( x_0 = n_0/N \) and parameters \( \lambda \neq \mu \).

(b) Show how the exponential long-time asymptotics crosses over to power-law asymptotics in the limit \( \lambda - \mu \to 0 \). (c) Find the dependence of the stationary population \( x(\infty) \) on \( \lambda, \mu, \gamma \).

(d) Plot \( x(t)/x_0 \) versus \( t \) for \( \mu = 1 \) and (i) various values of \( \lambda \) at fixed \( \gamma = 0.2 \) and (ii) various values of \( \gamma \) at fixed \( \lambda = 2 \). Interpret your results.

**Solution:**