[nex111] Effects of nonlinear death rates I: Malthus-Verhulst equation

Consider the master equation of the birth-death process with transition rates,

$$W(m|n) = n\lambda\delta_{m,n+1} + \left[n\mu + \frac{\gamma}{N}n(n-1)\right]\delta_{m,n-1}.$$

It describes a population with a linear birth rate, $n\lambda$, and a linear death rate, $n\mu$, as in [nex44]. To account for the unhealthy environment in crowded conditions ($n \simeq N$), a nonlinear death rate has been added. The nonlinearity suppresses the continued exponential growth found in [nex44] for the case $\lambda > \mu$ and stabilizes a stationary state.

(a) Construct the equations of motion for $\langle n(t)\rangle$ from the first jump moment as in [nex44]. Then neglect fluctuations by setting $\langle n^2(t)\rangle \simeq \langle n(t)\rangle^2$ and $\langle n(t)\rangle = Nx(t) + \mathrm{O}(\sqrt{N})$ to arrive (in leading order) at the Malthus-Verhulst equation,

$$\frac{dx}{dt} = (\lambda - \mu)x - \gamma x^2,$$

for the time-dependence of the (scaled) average population. Derive the analytic solution,

$$x(t) = \frac{x_0 e^{(\lambda-\mu)t}}{1 + \frac{\gamma x_0}{\lambda-\mu}\left[e^{(\lambda-\mu)t} - 1\right]},$$

pertaining to initial value $x_0 = n_0/N$ and parameters $\lambda \neq \mu$.

(b) Show how the exponential long-time asymptotics crosses over to power-law asymptotics in the limit $\lambda - \mu \to 0$. (c) Find the dependence of the stationary population $x(\infty)$ on λ, μ, γ.

(d) Plot $x(t)/x_0$ versus t for $\mu = 1$ and (i) various values of λ at fixed $\gamma = 0.2$ and (ii) various values of γ at fixed $\lambda = 2$. Interpret your results.

Solution: