[nex112] Populations with linear birth and death rates II

Consider the master equation

$$\frac{d}{dt}P(n,t) = \sum_m [W(n|m)P(m,t) - W(m|n)P(n,t)]$$

for the probability distribution $P(n,t)$ of the linear birth-death process with initial population n_0. It is specified by the transition rates

$$W(m|n) = n\lambda\delta_{m,n+1} + n\mu\delta_{m,n-1},$$

where λ and μ represent the birth and death rates, respectively, of individuals in some population.
(a) Convert the the master equation into a linear 1$^{\text{st}}$ order PDE for the generating function as follows:

$$\frac{\partial G}{\partial t} - (z-1)(\lambda z - \mu)\frac{\partial G}{\partial z} = 0, \quad G(z,t) \doteq \sum_{n=0}^{\infty} z^n P(n,t).$$

(b) Solve the PDE by using the method of characteristics:

$$\frac{1}{dt} = -\frac{(z-1)(\lambda z - \mu)}{dz} = \frac{0}{dG}.$$

The result reads

$$G(z,t) = \left(\frac{u(z,t)-\mu}{u(z,t)-\lambda}\right)^{n_0}, \quad u(z,t) = \frac{\lambda z - \mu}{z-1}e^{-(\lambda-\mu)t}.$$

(c) Simplify the expression of $G(z,t)$ for the special case $\lambda = \mu$.
(d) For the case $\lambda = \mu$ calculate mean $\langle\langle n(t)\rangle\rangle$ and variance $\langle\langle n^2(t)\rangle\rangle$ from derivatives of $G(z,t)$.

Solution: