[nex113] Modified linear birth rate I: stationarity

Consider a linear birth-death process with a modified birth rate,

$$W(m|n) = \lambda(n+1)\delta_{m,n+1} + \mu n\delta_{m,n-1},$$

to be used in the master equation. In [nex44] we had shown that the original linear birth rate $T_+(n) = n\lambda$ leads to the extinction of the population if $\lambda < \mu$.
(a) Use the recurrence relation of [nln17] to show that the modified linear birth rate $T_+(n) = (n+1)\lambda$ leads to a nonvanishing stationary distribution $P_s(n)$ for $\lambda < \mu$. Use the ratio $\gamma = \lambda/\mu$ as parameter of that distribution. Show that $P_s(n)$ is the Pascal distribution (see [nex22]).
(b) Determine the stationary values of $\langle n\rangle$ and $\langle\langle n^2\rangle\rangle$. Describe the differences between the long-time asymptotics of the original linear birth-death process with $\lambda = \mu$ as analyzed in [nex130] and the results obtained here for the case of modified birth rates in the limit $\mu - \lambda \to 0$.

Solution: