[nex114] Modified linear birth rate II: evolution of mean and variance

Consider a linear birth-death process with a modified birth rate to be used in the master equation:

$$W(m|n) = \lambda(n+1)\delta_{m,n+1} + \mu n\delta_{m,n-1}, \quad \lambda < \mu.$$

In [nex44] we had shown that the original linear birth rate $T_+(n) = n\lambda$ leads to the extinction of the population. In [nex113] we showed that the modified linear birth rate $T_+(n) = (n+1)\lambda$ leads to a nonvanishing stationary distribution $P_s(n)$: the Pascal distribution.
(a) Establish from the jump moments, $\alpha_l(n) = \sum_m (m-n)^l W(m|n)$, the equations of motion,

$$\frac{d}{dt}\langle n\rangle \doteq \langle \alpha_1(n)\rangle = (\lambda-\mu)\langle n\rangle + \lambda,$$

$$\frac{d}{dt}\langle n^2\rangle \doteq \langle \alpha_2(n)\rangle + 2\langle n\alpha_1(n)\rangle = 2(\lambda-\mu)\langle n^2\rangle + (3\lambda+\mu)\langle n(t)\rangle + \lambda,$$

for the first and second moment.
(b) Calculate the time evolution of mean $\langle\langle n(t)\rangle\rangle$ and variance $\langle\langle n^2(t)\rangle\rangle$ by solving these equations for initial conditions $\langle\langle n(0)\rangle\rangle = \langle\langle n^2(0)\rangle\rangle = 0$.

Solution: