[nex118] Mean-square displacement of Brownian particle III

Consider a Brownian particle of mass m constrained to move along a straight line. The particle experiences two forces: a drag force $-\gamma\dot{x}$ and a white-noise random force $f(t)$. Its motion is governed by the Langevin equation,

$$m\ddot{x} = -\gamma\dot{x} + f(t). \tag{1}$$

(a) Construct from (1) the linear ODE for the mean-square displacement,

$$m\frac{d^2}{dt^2}\langle x^2\rangle + \gamma\frac{d}{dt}\langle x^2\rangle = 2k_BT, \tag{2}$$

by using equipartition, $\frac{1}{2}m\langle\dot{x}^2\rangle = \frac{1}{2}k_BT$ and the fact that position and random force at the same instant are uncorrelated, $\langle xf(t)\rangle = 0$.
(b) Solve this ODE for initial conditions $d\langle x^2\rangle/dt|_0 = 0$ and $\langle x^2\rangle|_0 = 0$. Note that (2) is a first-order ODE for the variable $d\langle x^2\rangle/dt$.
(c) Identify the quadratic time-dependence of $\langle x^2\rangle$ in the ballistic regime, $t \ll m/\gamma$, and the linear time dependence in the diffusive regime, $t \gg m/\gamma$. Express the last result in terms of the diffusion constant by invoking Einstein's fluctuation-dissipation relation from [nln67].

Solution: