[nex119] Velocity correlation function of Brownian particle II

Consider the Langevin equation for the velocity of a Brownian particle of mass m constrained to move along a straight line,

$$m\frac{dv}{dt} = -\gamma v + f(t), \tag{1}$$

where γ is the damping constant and $f(t)$ is a white-noise random force, $\langle f(t)f(t')\rangle = I_f\delta(t-t')$, with intensity $I_f = 2k_BT\gamma$ in thermal equilibrium (see [nex55]).
(a) Convert the differential equation (1) into the algebraic relation,

$$-i\omega m\tilde{v}(\omega) = -\gamma\tilde{v}(\omega) + \tilde{f}(\omega), \tag{2}$$

between the Fourier amplitudes,

$$\tilde{v}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}v(t), \quad \tilde{f}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}f(t), \tag{3}$$

of the velocity and the random force, respectively.
(b) Apply the Wiener-Khintchine theorem (see [nln14]) to calculate from (2) and the above specifications of the random force the velocity spectral density and, via inverse Fourier transform, the velocity correlation function in thermal equilibrium:

$$S_{vv}(\omega) = \frac{2k_BT\gamma}{\gamma^2+\omega^2m^2}, \quad \langle v(t)v(t+\tau)\rangle = \frac{k_BT}{m}e^{-\gamma\tau/m}. \tag{4}$$

Solution: