[nex120] Velocity correlation function of Brownian particle III

The generalized Langevin equation for a particle of mass m constrained to move along a straight line,

$$m\frac{dv}{dt} = -\int_{-\infty}^{t} dt'\alpha(t-t')v(t') + f(t),$$

is known to produce the following expression for the spectral density of the velocity:

$$S_{vv}(\omega) = \frac{S_{ff}(\omega)}{|\hat{\alpha}(\omega) - i\omega m|^2}, \quad \hat{\alpha}(\omega) \doteq \int_0^{\infty} dt\, e^{i\omega t}\alpha(t), \quad S_{ff}(\omega) = 2k_BT\,\Re[\hat{\alpha}(\omega)],$$

where the relation between the random-force spectral density, $S_{ff}(\omega)$, and the Laplace-transformed attenuation function, $\hat{\alpha}(\omega)$, is dictated by the fluctuation-dissipation theorem.
The special case of Brownian motion (see [nex55], [nex119]) uses attenuation without memory: $\alpha(t-t') = 2\gamma\delta(t-t')\theta(t-t')$. Calculate the velocity correlation function, $\langle v(t)v(t')\rangle$, of the Brownian particle in thermal equilibrium from the above expression for $S_{vv}(\omega)$ via contour integration in the plane of complex ω.

Solution: