[nex122] Brownian harmonic oscillator II: position correlation function

The Brownian harmonic oscillator is specified by the Langevin-type equation, $m\ddot{x}+\gamma\dot{x}+kx = f(t)$, where m is the mass of the particle, γ represents attenuation without memory, $k = m\omega_0^2$ is the spring constant, and $f(t)$ is a white-noise random force.

(a) Start from the result $S_{xx}(\omega) = 2\gamma k_B T/[m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]$ for the spectral density of the position coordinate as calculated in [nex121] to derive the position correlation function

$$\langle x(t)x(0)\rangle \doteq \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} S_{xx}(\omega) = \begin{cases} \frac{k_B T}{m\omega_0^2}\, e^{-\frac{\gamma}{2m}t}\left[\cos\omega_1 t + \frac{\gamma}{2m\omega_1}\sin\omega_1 t\right] \\ \frac{k_B T}{m\omega_0^2}\, e^{-\frac{\gamma}{2m}t}\left[1 + \frac{\gamma}{2m}t\right] \\ \frac{k_B T}{m\omega_0^2}\, e^{-\frac{\gamma}{2m}t}\left[\cosh\Omega_1 t + \frac{\gamma}{2m\Omega_1}\sinh\Omega_1 t\right] \end{cases}$$

for the cases $\omega_1 = \sqrt{\omega_0^2 - \gamma^2/4m^2} > 0$ (underdamped), $\omega_0^2 = \gamma^2/4m^2$ (critically damped), and $\Omega_1 = \sqrt{\gamma^2/4m^2 - \omega_0^2} > 0$ (overdamped), respectively.

(b) Plot $S_{xx}(w)$ versus ω/ω_0 and $\langle x(t)x(0)\rangle m\omega_0^2/k_B T$ versus $\omega_0 t$ with three curves in each frame, one for each case. Use Mathematica for both parts and supply a copy of the notebook.

Solution: