[nex123] Brownian harmonic oscillator III: contour integrals

The generalized Langevin equation for the Brownian harmonic oscillator,

$$m\frac{dx}{dt} + \int_{-\infty}^{t} dt'\alpha(t-t')x(t') = \frac{1}{\omega_0}f(t), \quad \alpha(t) = m\omega_0^2 e^{-(\gamma/m)t}, \tag{1}$$

where $\alpha(t)$ is the attenuation function, $m\omega_0^2$ the spring constant, and $f(t)$ a correlated-noise random force, is known to produce the following expression for the spectral density of the position coordinate:

$$S_{xx}(\omega) = \frac{S_{ff}(\omega)/\omega_0^2}{|\hat{\alpha}(\omega) - i\omega m|^2}, \quad \hat{\alpha}(\omega) = \int_0^{\infty} dt\, e^{i\omega t}\alpha(t), \quad S_{ff}(\omega) = 2k_BT\Re[\hat{\alpha}(\omega)], \tag{2}$$

where the relation between the random-force spectral density, $S_{ff}(\omega)$, and the Laplace-transformed attenuation function, $\hat{\alpha}(\omega)$, is dictated by the fluctuation-dissipation relation introduced in [nln72].
(a) Calculate $S_{ff}(\omega)$ or restate the result used in [nex129] and determine its singularity structure.
(b) Evaluat $S_{xx}(\omega)$ and identify its singularity structure for the cases (i) $\gamma/2m < \omega_0$ (underdamped), (ii) $\gamma/2m = \omega_0$ (critically damped), and (iii) $\gamma/2m > \omega_0$ (overdamped).
(c) Calculate

$$\langle x(t)x(0)\rangle \doteq \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} S_{xx}(\omega) \tag{3}$$

via contour integration for the cases (i)-(iii) and check the results against those obtained in [nex122].

Solution: