[nex130] Populations with linear birth and death rates III

Consider the master equation

$$\frac{d}{dt}P(n,t) = \sum_m [W(n|m)P(m,t) - W(m|n)P(n,t)]$$

for the probability distribution $P(n,t)$ of the linear birth-death process with initial population n_0. It is specified by the transition rates

$$W(m|n) = n\lambda\delta_{m,n+1} + n\mu\delta_{m,n-1},$$

where λ and μ represent the birth and death rates, respectively, of individuals in some population. In [nex112] we have determined the following expression for the generating function pertaining to the special case $\lambda = \mu$ of equal birth and death rates:

$$G(z,t) \doteq \sum_{n=0}^{\infty} z^n P(n,t) = \left(\frac{\lambda(z-1)t - z}{\lambda(z-1)t - 1}\right)^{n_0}.$$

(a) Expand the generating function in powers of z for the special case $n_0 = 1$ to derive analytic expressions for the probabilities $P(n,t)$.
(b) Verify the normalization condition $\sum_n P(n,t) = 1$.
(c) Plot $P(n,t)$ versus λt for $n = 0, 1, \ldots, 4$ (five curves).
(d) Find the time t_n where $P(n,t)$ reaches its maximum value.
(e) Discuss the compatibility of the paradoxical results (i) $\lim_{t\to\infty} P(0,t) = 1$ (certainty of death) and from [nex44] (ii) $\langle n(t)\rangle = n_0 = 1$, (iii) $\lim_{t\to\infty}\langle\langle n^2(t)\rangle\rangle \to \infty$ (persistent signs of life and uncertainty).
(f) Explore the determination of analytic expressions of $P(n,t)$ for $n_0 = 2, 3, \ldots$ or for generic n_0.

Solution: