[nex19] Robust probability distributions

Consider two independent stochastic variables X_1 and X_2, each specified by the same probability distribution $P_X(x)$. Show that if $P_X(x)$ is either a Gaussian, a Lorentzian, or a Poisson distribution,

$$\text{(i)} \quad P_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-x^2/2\sigma^2}, \qquad \text{(ii)} \quad P_X(x) = \frac{1}{\pi}\frac{a}{x^2+a^2}, \qquad \text{(iii)} \quad P_X(x=n) = \frac{a^n}{n!}\, e^{-a}.$$

then the probability distribution $P_Y(y)$ of the stochastic variable $Y = X_1 + X_2$ is also a Gaussian, a Lorentzian, or a Poisson distribution, respectively. What property of the characteristic function $\Phi_X(k)$ guarantees the robustness of $P_X(x)$?

Solution: