[nex28] Master equation for a continuous random variable.

The master equation,

$$\frac{\partial}{\partial t} P(x|x_0;t) = \int dx' [W(x|x')P(x'|x_0;t) - W(x'|x)P(x|x_0;t)],$$

describes the time evolution of probability distributions for pure jump processes. The rate at which $P(x|x_0;t)$ evolves in time is governed by two contributions: a positive contribution from jumps $x' \to x$ taking place at the rate $W(x|x')$ and a negative contribution from jumps $x \to x'$ taking place at the rate $W(x'|x)$. The goal of this exercise is to derive the master equation from the ansatz

$$P(x|x';\Delta t) = \Delta t\, W(x|x') + \delta(x-x')[1 - \Delta t \int dx'' W(x''|x')],$$

for the conditional probability distribution assumed to hold for infinitesimal time intervals Δt. In this ansatz, the first term represents the probability density for transitions $x' \to x \neq x'$ during Δt and the second term the probability density for no transitions occurring within Δt.

Hint: Start from the (integral) Chapman-Kolmogrov equation for $P(x|x_0;t)$ and construct the partial time derivative via $\lim_{\Delta t \to 0}[P(x|x_0;t+\Delta t) - P(x|x_0;t)]/\Delta t$. Then substitute the ansatz.

Solution: