[nex30] Equations of motion for mean value and variance.

Consider the Fokker-Planck equation for a stochastic process,

$$\frac{\partial}{\partial t}\,P(x,t|x_0) = -\frac{\partial}{\partial x}\,[A(x)P(x,t|x_0)] + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\,[B(x)P(x,t|x_0)],$$

where x_0 is the initial value of all sample paths, implying $P(x,0|x_0) = \delta(x-x_0)$. Use the equations of motion,

$$\frac{d}{dt}\,\langle x\rangle = \langle A(x)\rangle, \qquad \frac{d}{dt}\,\langle x^2\rangle = \langle B(x)\rangle + 2\langle xA(x)\rangle,$$

to calculate the time-dependence of the mean value $\langle x\rangle$ and the variance $\langle\langle x^2\rangle\rangle$ for two processes with initial conditions as dictated by $P(x,0|x_0) = \delta(x-x_0)$:
(i) Uniform drift and diffusion process: $A(x) = v$, $B(x) = 2D$.
(ii) Ornstein-Uhlenbeck process: $A(x) = -\kappa x$, $B(x) = \gamma$.

Solution: