[nex31] Fokker-Planck equation for Ornstein-Uhlenbeck process.

Consider the Ornstein -Uhlenbeck process as specified by the Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(\kappa x P\right) + \frac{1}{2}\gamma \frac{\partial^2 P}{\partial x^2}, \tag{1}$$

for the conditional probability distribution $P(x,t|x_0)$, where x_0 specifies the intial value of all sample paths: $P(x,0|x_0) = \delta(x - x_0)$.
(a) Derive from the 2^{nd} order PDE (1) the 1^{st} order PDE for the characteristic function:

$$\frac{\partial \Phi}{\partial t} + \kappa s \frac{\partial}{\partial s}\Phi(s,t) = -\frac{1}{2}\gamma s^2 \Phi(s,t), \quad \Phi(s,t) \doteq \int_{-\infty}^{+\infty} dx\, e^{isx} P(x,t|x_0). \tag{2}$$

(b) Solve (2) by the method of characteristics,

$$\frac{1}{dt} = \frac{\kappa s}{ds} = -\frac{\frac{1}{2}\gamma s^2 \Phi}{d\Phi}. \tag{3}$$

(c) Infer from the solution $\Phi(s,t)$ an explicit expression for $P(x,t|x_0)$.

Solution: