[nex39] Regression theorem for autocorrelation functions.

The regression theorem for autocorrelation functions of a Markov process reads

$$\langle X(t)X(t_0)\rangle = \int dx \int dx' x x_0 P(x,t;x_0,t_0) = \int dx_0 \langle X(t)|[x_0,t_0]\rangle x_0 P(x_0,t),$$

where $\langle X(t)|[x_0,t_0]\rangle \equiv \int dx\, xP(x,t|x_0,t_0)$ is the definition of a conditional average.
(a) Show that if $\lim_{t_0\to-\infty} P(x,t|x_0,t_0) = P_S(x)$ independent of t, x_0, then the autocorrelation function in a stationary process is

$$\langle X(t)X(t')\rangle_S = \lim_{t_0\to-\infty} \langle X(t)X(t')|[x_0,t_0]\rangle = \int dx \int dx' x x' P(x,t|x',t')P_S(x').$$

(b) Apply the regression theorem to calculate $\langle X(t)X(t')\rangle_S$ for the Ornstein-Uhlenbeck process at stationarity.

Solution: