[nex44] Populations with linear birth and death rates I

Consider the master equation

$$\frac{d}{dt}P(n,t) = \sum_m [W(n|m)P(m,t) - W(m|n)P(n,t)]$$

for the probability distribution $P(n,t)$ of the linear birth-death process. It is specified by the transition rates

$$W(m|n) = n\lambda\delta_{m,n+1} + n\mu\delta_{m,n-1},$$

where λ and μ represent the birth and death rates, respectively, of individuals in some population.
(a) Determine the jump moments $\alpha_l(m) = \sum_n (n-m)^l W(n|m)$ for $l = 1, 2$.
(b) Calculate the time evolution of the mean value $\langle n \rangle$ and the variance $\langle\langle n^2 \rangle\rangle$ for the initial condition $P(n,0) = \delta_{n,n_0}$ by solving the equations of motion for the expectation values,

$$\frac{d}{dt}\langle n \rangle = \langle \alpha_1(n) \rangle, \qquad \frac{d}{dt}\langle n^2 \rangle = \langle \alpha_2(n) \rangle + 2\langle n\alpha_1(n) \rangle,$$

introduced in [nln59].
(c) Plot $\langle\langle n^2 \rangle\rangle$ versus t for three cases with (i) $\lambda > \mu$, (ii) $\lambda = \mu$, and (iii) $\lambda < \mu$. Interpret the shape of each curve.

Solution: