[nex48] Air in leaky tank I: generating function

At time $t = 0$ a tank of volume V contains n_0 molecules of air (disregarding chemical distinctions). The tank has a tiny leak and exchanges molecules with the environment, which has a constant density ρ of air molecules.
(a) Set up the master equation for the probability distribution $P(n,t)$ under the assumption that a molecule leaves the tank with probability $(n/V)dt$ and enters the tank with probability ρdt, implying transition rates $W(m|n) = \rho\delta_{m,n+1} + (n/V)\delta_{m,n-1}$.
(b) Derive the following linear partial differential equation (PDE) for the generating function $G(z,t)$ from that master equation:

$$\frac{\partial G}{\partial t} + \frac{z-1}{V}\frac{\partial G}{\partial z} = \rho(z-1)G, \quad G(z,t) \doteq \sum_{n=0}^{\infty} z^n P(n,t).$$

(c) Solve the PDE by the method of characteristics,

$$\frac{1}{dt} = \frac{z-1}{Vdz} = \frac{\rho(z-1)G}{dG},$$

to obtain the result

$$G(z,t) = e^{V\rho(z-1)\left[1-e^{-t/V}\right]}\left[e^{-t/V}(z-1)+1\right]^{n_0}$$

for the nonequilibrium state.
(d) Find the characteristic function $G(z,\infty)$ for the equilibrium situation.
(e) If we set n_0/V (density inside) equal to ρ (density outside), the generating function still depends on time. Explain the reason.
(f) Show that for $n_0/V = \rho$ the function $G(z,t)$ at arbitrary t converges, as $\rho V \to \infty$, toward the stationary result determined in (e).

Solution: