Air in leaky tank I: generating function

At time $t = 0$ a tank of volume $V$ contains $n_0$ molecules of air (disregarding chemical distinctions). The tank has a tiny leak and exchanges molecules with the environment, which has a constant density $\rho$ of air molecules.

(a) Set up the master equation for the probability distribution $P(n, t)$ under the assumption that a molecule leaves the tank with probability $(n/V)dt$ and enters the tank with probability $\rho dt$, implying transition rates $W(m|n) = \rho \delta_{m,n+1} + (n/V) \delta_{m,n-1}$.

(b) Derive the following linear partial differential equation (PDE) for the generating function $G(z, t)$ from that master equation:

$$\frac{\partial G}{\partial t} + \frac{z-1}{V} \frac{\partial G}{\partial z} = \rho (z-1) G, \quad G(z, t) = \sum_{n=0}^{\infty} z^n P(n, t).$$

(c) Solve the PDE by the method of characteristics,

$$\frac{1}{dt} = \frac{z-1}{V dz} = \frac{\rho (z-1) G}{dG},$$

to obtain the result

$$G(z, t) = e^{V \rho (z-1)[1-e^{-t/V}] \left[ e^{-t/V}(z-1) + 1 \right]^{n_0}}$$

for the nonequilibrium state.

(d) Find the characteristic function $G(z, \infty)$ for the equilibrium situation.

(e) If we set $n_0/V$ (density inside) equal to $\rho$ (density outside), the generating function still depends on time. Explain the reason.

(f) Show that for $n_0/V = \rho$ the function $G(z, t)$ at arbitrary $t$ converges, as $\rho V \to \infty$, toward the stationary result determined in (e).

Solution: