[nex51] Effects of nonlinear death rate II: stationarity and fluctuations

Consider the master equation of the birth-death process with transition rates

$$W(m|n) = (n+1)\lambda\delta_{m,n+1} + \left[n\mu + \frac{\gamma}{N}n(n-1)\right]\delta_{m,n-1}.$$

It describes a population with a linear birth rate, $(n+1)\lambda$, and a linear death rate, $n\mu$. To account for the unhealthy environment under crowded circumstances ($n \simeq N$), a nonlinear death rate has been added to the process. Use the recurrence relation, $P_s(n) = [T_+(n-1)/T_-(n)]P_s(n-1)$ for the stationary distribution $P_s(n)$ derived in [nln17] from the detailed balance condition for the following tasks. Consider a system that easily accommodates $N = 20$ individuals of some population with fixed (linear) death rate $\mu = 1$, fixed birth rate $\lambda = 1.5$, and variable environmental factor γ.
(a) Compute the mean $\langle n\rangle$ and the variance $\langle\langle n^2\rangle\rangle$ for $\gamma = 0.2$.
(b) Plot the distribution $P(n)$ versus n for $\gamma = 0.2, 0.4, \ldots, 1.0$ in the same diagram.
(c) Plot the mean $\langle n\rangle$ across the range $0.2 < \gamma < 1$ and compare the result with the function $Nx(t)$ derived in [nex111] from the Malthus-Verhulst equation, which ignores fluctuations.
(d) Plot the variance $\langle\langle n^2\rangle\rangle$ across the range $0.2 < \gamma < 1$.

Solution: