[nex58] Brownian harmonic oscillator IV: velocity correlations

The Brownian harmonic oscillator is specified by the Langevin-type equation,

$$m\ddot{x} + \gamma\dot{x} + kx = f(t), \tag{1}$$

where m is the mass of the particle, γ represents attenuation without memory, $k = m\omega_0^2$ is the spring constant, and $f(t)$ is a white-noise random force.
(a) Find the velocity spectral density by proving the relation

$$S_{vv}(\omega) = \omega^2 S_{xx}(\omega) \tag{2}$$

and using the result from [nex121] for the position spectral density $S_{xx}(\omega)$.
(b) Find the velocity correlation function by proving the relation

$$\langle v(t)v(0)\rangle = -\frac{d^2}{dt^2}\langle x(t)x(0)\rangle \tag{3}$$

and using the result from [nex122] for the position correlation function. Distinguish the cases (i) $\omega_1 = \sqrt{\omega_0^2 - \gamma^2/4m^2} > 0$ for underdamped motion, (ii) $\omega_0^2 = \gamma^2/4m^2$ for critically damped motion, and (iii) $\Omega_1 = \sqrt{\gamma^2/4m^2 - \omega_0^2} > 0$ for overdamped motion.
(c) Plot $S_{vv}(\omega)$ versus ω/ω_0 and $\langle v(t)v(0)\rangle m/k_BT$ versus $\omega_0 t$ with three curves in each frame, one for each case.

Solution: