[nex59] Brownian harmonic oscillator V: formal solution for velocity

Convert the Langevin-type equation, $m\ddot{x}+\gamma\dot{x}+kx = f(t)$, for the overdamped Brownian harmonic oscillator with mass m, damping constant γ, spring constant $k = m\omega_0^2$, and white-noise random force $f(t)$ into a second-order ODE for the stochastic variable $v(t)$. Then show that

$$v(t) = v_0 e^{-\Gamma t} c(t) - \frac{\omega_0^2}{\Omega_1} x_0 e^{-\Gamma t} \sinh \Omega_1 t + \frac{1}{m} \int_0^t dt' f(t') e^{-\Gamma(t-t')} c(t-t')$$

with $\Gamma = \gamma/2m$, $\Omega_1 = \sqrt{\Gamma^2 - \omega_0^2}$, $c(t) = \cosh \Omega_1 t - (\Gamma/\Omega_1) \sinh \Omega_1 t$ is a formal solution for initial conditions $x(0) = x_0$ and $v(0) = v_0$.

Solution: