[nex66] Linear response of classical relaxator.

The classical relaxator is defined by the equation of motion,

\[ \dot{x} + \frac{1}{\tau_0} x = a(t), \]  

(1)

where \( \tau_0 \) represents a relaxation time and \( a(t) \) a weak periodic perturbation. The (linear) response function is extracted from the relation

\[ \langle x(t) \rangle - \langle x \rangle_0 = \int_{-\infty}^{t} dt' \tilde{\chi}_{xx}(t-t')a(t'), \]

(2)

where \( x(t) \) is the solution of (1).

(a) Solve (1) formally as in [nex53] and compare the result with (2) to show that the response function must be

\[ \tilde{\chi}_{xx}(t) = e^{-t/\tau_0} \theta(t). \]

(3)

(b) Calculate the generalized susceptibility \( \chi_{xx}(\omega) \) via Fourier analysis of (1) as in [nex119]. Show that the Fourier transform of (3) yields the same result, namely

\[ \chi_{xx}(\omega) = \frac{\tau_0}{1 - i\omega \tau_0}. \]

(4)

(c) Extract from \( \chi_{xx}(\omega) \) its reactive part \( \chi'_{xx}(\omega) \) and its dissipative part \( \chi''_{xx}(\omega) \) as prescribed in [nln30] and verify their symmetry properties.

(d) Use the (classical) fluctuation-dissipation theorem from [nln39] to infer the spectral density \( \Phi_{xx}(\omega) \) from the dissipation function \( \chi''_{xx}(\omega) \).

(e) Retrieve from the generalized susceptibility (4) the response function (3) via inverse Fourier transform carried out as a contour integral.

(f) Retrieve \( \chi'_{xx}(\omega) \) from \( \chi''_{xx}(\omega) \) and vice versa via a numerical principal-value integration of the Kramers-Kronig relations as stated in [nln37]. Use \( \tau = 1 \) and consider the interval \(-2 \leq \omega \leq 2\).

Plot the curves obtained via integration for comparison with the analytic expressions. Integrate over the intervals \(-\infty < \omega' < \omega - \epsilon \) and \( \omega + \epsilon < \omega' < +\infty \) with \( 0 < \epsilon \ll 1 \).

Solution: