[nex66] Linear response of classical relaxator.

The classical relaxator is defined by the equation of motion,

$$\dot{x} + \frac{1}{\tau_0}\, x = a(t), \tag{1}$$

where τ_0 represents a relaxation time and $a(t)$ a weak periodic perturbation. The (linear) response function is extracted from the relation

$$\langle x(t)\rangle - \langle x\rangle_0 = \int_{-\infty}^{t} dt' \tilde{\chi}_{xx}(t-t')a(t'), \tag{2}$$

where $x(t)$ is the solution of (1).
(a) Solve (1) formally as in [nex53] and compare the result with (2) to show that the response function must be

$$\tilde{\chi}_{xx}(t) = e^{-t/\tau_0}\theta(t). \tag{3}$$

(b) Calculate the generalized susceptibility $\chi_{xx}(\omega)$ via Fourier analysis of (1) as in [nex119]. Show that the Fourier transform of (3) yields the same result, namely

$$\chi_{xx}(\omega) = \frac{\tau_0}{1 - i\omega\tau_0}. \tag{4}$$

(c) Extract from $\chi_{xx}(\omega)$ its reactive part $\chi'_{xx}(\omega)$ and its dissipative part $\chi''_{xx}(\omega)$ as prescribed in [nln30] and verify their symmtry properties.
(d) Use the (classical) fluctuation-dissipation theorem from [nln39] to infer the spectral density $\Phi_{xx}(\omega)$ from the dissipation function $\chi''_{xx}(\omega)$.
(e) Retrieve from the generalized susceptibility (4) the response function (3) via inverse Fourier transform carried out as a contour integral.
(f) Retrieve $\chi'_{xx}(\omega)$ from $\chi''_{xx}(\omega)$ and vice versa via a numerical principal-value integration of the Kramers-Kronig relations as stated in [nln37]. Use $\tau = 1$ and consider the interval $-2 \leq \omega \leq 2$. Plot the curves obtained via integration for comparison with the analytic expressions. Integrate over the intervals $-\infty < \omega' < \omega - \epsilon$ and $\omega + \epsilon < \omega' < +\infty$ with $0 < \epsilon \ll 1$.

Solution: