Non-differentiability of sample paths

The non-differentiability of sample paths of stochastic processes can be investigated by calculating the probability that for any value of $x$ the slope $\lim_{\Delta t \to 0} |\Delta x/\Delta t|$ is greater than any chosen value $\kappa > 0$. Calculate

$$\lim_{\Delta t \to 0} P[|\Delta x/\Delta t| > \kappa] = \int_{|x-x_0|>\kappa \Delta t} dx P(x|x_0; \Delta t)$$

for the diffusion process and the Cauchy process,

$$P(x|x_0; \Delta t) = \frac{1}{\sqrt{4\pi D \Delta t}} \exp \left( -\frac{(x-x_0)^2}{4D \Delta t} \right), \quad P(x|x_0; \Delta t) = \frac{\Delta t}{\pi (x-x_0)^2 + (\Delta t)^2}.$$

For comparison, calculate the same probability for the deterministic drift process,

$$P(x|x_0; \Delta t) = \delta(x - x_0 - v \Delta t).$$

Draw your own conclusions from the results. State a necessary criterion for a sample path to be differentiable at a given value of $x$. Which process satisfies your criterion?

Solution: