[nex99] Non-differentiability of sample paths

The non-differentiability of sample paths of stochastic processes can be investigated by calculating the probability that for any value of x the slope $\lim_{\Delta t\to 0}|\Delta x/\Delta t|$ is greater than any chosen value $\kappa > 0$. Calculate

$$\lim_{\Delta t\to 0}\mathcal{P}[|\Delta x/\Delta t| > \kappa] = \int_{|x-x_0|>\kappa\Delta t} dx\, P(x|x_0;\Delta t)$$

for the diffusion process and the Cauchy process,

$$P(x|x_0;\Delta t) = \frac{1}{\sqrt{4\pi D\Delta t}}\exp\left(-\frac{(x-x_0)^2}{4D\Delta t}\right), \qquad P(x|x_0;\Delta t) = \frac{1}{\pi}\frac{\Delta t}{(x-x_0)^2+(\Delta t)^2}.$$

For comparison, calculate the same probability for the deterministic drift process,

$$P(x|x_0;\Delta t) = \delta(x-x_0-v\Delta t).$$

Draw your own conclusions from the results. State a necessary criterion for a sample path to be differentiable at a given value of x. Which process satisfies your criterion?

Solution: