If your problem requires a series of related binomial coefficients, a good idea is to use recurrence relations, for example
\[
\binom{n+1}{k} = \frac{n+1}{n-k+1} \binom{n}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{6.1.7}
\]
\[
\binom{n}{k+1} = \frac{n-k}{k+1} \binom{n}{k}
\]

Finally, turning away from the combinatorial functions with integer valued arguments, we come to the beta function,
\[
B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt
\]
which is related to the gamma function by
\[
B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{6.1.8}
\]

hence

```fortran
FUNCTION beta(z,w)
REAL beta,w,z
C USES gammln

Returns the value of the beta function B(z,w).
REAL gammln
beta=exp(gammln(z)+gammln(w)-gammln(z+w))
return
END
```

CITED REFERENCES AND FURTHER READING:

### 6.2 Incomplete Gamma Function, Error Function, Chi-Square Probability Function, Cumulative Poisson Function

The incomplete gamma function is defined by
\[
P(a, x) \equiv \frac{\gamma(a, x)}{\Gamma(a)} \equiv \frac{1}{\Gamma(a)} \int_0^x e^{-t}t^{a-1}dt \quad (a > 0) \tag{6.2.1}
\]
The incomplete gamma function $P(a, x)$ for four values of $a$.

It has the limiting values

$$P(a, 0) = 0 \quad \text{and} \quad P(a, \infty) = 1 \quad (6.2.2)$$

The incomplete gamma function $P(a, x)$ is monotonic and (for $a$ greater than one or so) rises from “near-zero” to “near-unity” in a range of $x$ centered on about $a - 1$, and of width about $\sqrt{a}$ (see Figure 6.2.1).

The complement of $P(a, x)$ is also confusingly called an incomplete gamma function,

$$Q(a, x) \equiv 1 - P(a, x) = \frac{\Gamma(a, x)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_x^\infty e^{-t} t^{a-1} dt \quad (a > 0) \quad (6.2.3)$$

It has the limiting values

$$Q(a, 0) = 1 \quad \text{and} \quad Q(a, \infty) = 0 \quad (6.2.4)$$

The notations $P(a, x), \gamma(a, x)$, and $\Gamma(a, x)$ are standard; the notation $Q(a, x)$ is specific to this book.

There is a series development for $\gamma(a, x)$ as follows:

$$\gamma(a, x) = e^{-x} x^a \sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a + 1 + n)} x^n \quad (6.2.5)$$

One does not actually need to compute a new $\Gamma(a + 1 + n)$ for each $n$; one rather uses equation (6.1.3) and the previous coefficient.
A continued fraction development for $\Gamma(a, x)$ is

$$\Gamma(a, x) = e^{-x}x^a \left( \frac{1}{x + a} - \frac{1}{x + 1} + \frac{2}{x + 2} \cdots \right) \quad (x > 0) \quad (6.2.6)$$

It is computationally better to use the even part of (6.2.6), which converges twice as fast (see §5.2):

$$\Gamma(a, x) = e^{-x}x^a \left( \frac{1}{x + 1 - a} - \frac{1 - a}{x + 3 - a} + \frac{2 - a}{x + 5 - a} \cdots \right) \quad (x > 0) \quad (6.2.7)$$

It turns out that (6.2.5) converges rapidly for $x$ less than about $a + 1$, while (6.2.6) or (6.2.7) converges rapidly for $x$ greater than about $a + 1$. In these respective regimes each requires at most a few times $\sqrt{a}$ terms to converge, and this many only near $x = a$, where the incomplete gamma functions are varying most rapidly. Thus (6.2.5) and (6.2.7) together allow evaluation of the function for all positive $a$ and $x$. An extra dividend is that we never need compute a function value near zero by subtracting two nearly equal numbers. The higher-level functions that return $P(a, x)$ and $Q(a, x)$ are

```fortran
FUNCTION gammp(a,x)
REAL a,gammp,x
C USES gcf,gser
 Returns the incomplete gamma function $P(a, x)$.
REAL gammcf,gamser,gln
if(x.lt.0..or.a.le.0.)pause 'bad arguments in gammp'
if(x.lt.a+1.)then
  Use the series representation.
  call gser(gamser,a,x,gln)
  gammp=gamser
else
  Use the continued fraction representation.
  call gcf(gammcf,a,x,gln)
  gammp=1.-gammcf
endif
return
END

FUNCTION gammq(a,x)
REAL a,gammq,x
C USES gcf,gser
 Returns the incomplete gamma function $Q(a, x) \equiv 1 - P(a, x)$.
REAL gammcf,gamser,gln
if(x.lt.0..or.a.le.0.)pause 'bad arguments in gammq'
if(x.lt.a+1.)then
  Use the series representation.
  call gser(gamser,a,x,gln)
  gammq=1.-gamser
else
  Use the continued fraction representation.
  call gcf(gammcf,a,x,gln)
  gammq=gammcf
endif
return
END
```
The argument $gln$ is returned by both the series and continued fraction procedures containing the value $\ln \Gamma(a)$; the reason for this is so that it is available to you if you want to modify the above two procedures to give $\gamma(a,x)$ and $\Gamma(a,x)$, in addition to $P(a,x)$ and $Q(a,x)$ (cf. equations 6.2.1 and 6.2.3).

The procedures $gser$ and $gcf$ which implement (6.2.5) and (6.2.7) are

SUBROUTINE gser(gamser,a,x,gln)
INTEGER ITMAX
REAL a,gamser,gln,x,EPS
PARAMETER (ITMAX=100,EPS=3.e-7)
C USES gammln
Returns the incomplete gamma function $P(a,x)$ evaluated by its series representation as gamser. Also returns $\ln \Gamma(a)$ as gln.
INTEGER n
REAL ap,del,sum,gammln
gln=gammln(a)
if(x.le.0.)then
  if(x.lt.0.)pause 'x < 0 in gser'
  gamser=0.
  return
endif
ap=a
sum=1./a
del=sum
do n=1,ITMAX
  ap=ap+1.
  del=del*x/ap
  sum=sum+del
  if(abs(del).lt.abs(sum)*EPS)goto 1
enddo
1 pause 'a too large, ITMAX too small in gser'
  gamser=sum*exp(-x+a*log(x)-gln)
return
END

SUBROUTINE gcf(gammcf,a,x,gln)
INTEGER ITMAX
REAL a,gammcf,gln,x,EPS,FPMIN
PARAMETER (ITMAX=100,EPS=3.e-7,FPMIN=1.e-30)
C USES gammln
Returns the incomplete gamma function $Q(a,x)$ evaluated by its continued fraction representation as gammcf. Also returns $\ln \Gamma(a)$ as gln.
Parameters: ITMAX is the maximum allowed number of iterations; EPS is the relative accuracy; FPMIN is a number near the smallest representable floating-point number.
INTEGER i
REAL an,b,c,d,del,h,gammln
b=x+1.-a

Set up for evaluating continued fraction by modified Lentz's method (§5.2) with $b_0 = 0$.
gamln=gammln(a)
c=1./FPMIN
d=1./b
h=d
do i=1,ITMAX
  an=-i*(i-a)
  b=b+2.
  d=an*d+b
  if(abs(d).lt.FPMIN)d=FPMIN
  c=b+an/c
  if(abs(c).lt.FPMIN)c=FPMIN
  d=1./d
  del=d/c

Iterate to convergence.
6.2 Incomplete Gamma Function

h=h*del
   if(abs(del-1.).lt.EPS)goto 1
dendo:
   pause 'a too large, ITMAX too small in gcf'
1 gammcf=exp(-x+a*log(x)-gln)*h       Put factors in front.
return
END

Error Function

The error function and complementary error function are special cases of the incomplete gamma function, and are obtained moderately efficiently by the above procedures. Their definitions are

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt \]  
(6.2.8)

and

\[ \text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt \]  
(6.2.9)

The functions have the following limiting values and symmetries:

\[ \text{erf}(0) = 0 \quad \text{erf}(\infty) = 1 \quad \text{erf}(-x) = -\text{erf}(x) \]  
(6.2.10)

\[ \text{erfc}(0) = 1 \quad \text{erfc}(\infty) = 0 \quad \text{erfc}(-x) = 2 - \text{erfc}(x) \]  
(6.2.11)

They are related to the incomplete gamma functions by

\[ \text{erf}(x) = P\left(\frac{1}{2}, x^2\right) \quad (x \geq 0) \]  
(6.2.12)

and

\[ \text{erfc}(x) = Q\left(\frac{1}{2}, x^2\right) \quad (x \geq 0) \]  
(6.2.13)

Hence we have

FUNCTION erf(x)
REAL erf,x
C USES gammp
   Returns the error function erf(x).
REAL gammp
   if(x.lt.0.)then
      erf=gammp(.5,x**2)
   else
      erf=gammp(.5,x**2)
   endif
return
END
FUNCTION erfc(x)
REAL erfc, x
C USES gammp, gammq

Returns the complementary error function erfc(x).
REAL gammp, gammq
if (x.lt.0.) then
  erfc = 1. + gammp(.5, x**2)
else
  erfc = gammq(.5, x**2)
endif
return
END

If you care to do so, you can easily remedy the minor inefficiency in erf and erfc, namely that \( \Gamma(0.5) = \sqrt{\pi} \) is computed unnecessarily when gammp or gammq is called. Before you do that, however, you might wish to consider the following routine, based on Chebyshev fitting to an inspired guess as to the functional form:

FUNCTION erfcc(x)
REAL erfcc, x

Returns the complementary error function erfc(x) with fractional error everywhere less than \( 1.2 \times 10^{-7} \).
REAL t, z
z = abs(x)
t = 1./(1.+0.5*z)
erfcc = t * exp(-z*z - 1.26551223 + t*(1.00002368 + t*(0.37409196 + t*(0.09678418 + t*(-0.18628806 + t*(-0.27886807 + t*(-1.13520398 + t*(0.48851587 + t*(-0.82215223 + t*1.17087277)))))))))
if (x.lt.0.) erfcc = 2. - erfcc
return
END

There are also some functions of two variables that are special cases of the incomplete gamma function:

**Cumulative Poisson Probability Function**

\( P_x(< k) \), for positive \( x \) and integer \( k \geq 1 \), denotes the cumulative Poisson probability function. It is defined as the probability that the number of Poisson random events occurring will be between 0 and \( k - 1 \) inclusive, if the expected mean number is \( x \). It has the limiting values

\[
P_x(< 1) = e^{-x} \quad P_x(< \infty) = 1
\]

(6.2.14)

Its relation to the incomplete gamma function is simply

\[
P_x(< k) = Q(k, x) = \text{gammq}(k, x)
\]

(6.2.15)
Chi-Square Probability Function

\( P(\chi^2 | \nu) \) is defined as the probability that the observed chi-square for a correct model should be less than a value \( \chi^2 \). (We will discuss the use of this function in Chapter 15.) Its complement \( Q(\chi^2 | \nu) \) is the probability that the observed chi-square will exceed the value \( \chi^2 \) by chance even for a correct model. In both cases \( \nu \) is an integer, the number of degrees of freedom. The functions have the limiting values

\[
P(0 | \nu) = 0 \quad P(\infty | \nu) = 1 \quad (6.2.16)
\]
\[
Q(0 | \nu) = 1 \quad Q(\infty | \nu) = 0 \quad (6.2.17)
\]

and the following relation to the incomplete gamma functions,

\[
P(\chi^2 | \nu) = P\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) = \text{gammq}\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) \quad (6.2.18)
\]
\[
Q(\chi^2 | \nu) = Q\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) = \text{gammq}\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) \quad (6.2.19)
\]

CITED REFERENCES AND FURTHER READING:


6.3 Exponential Integrals

The standard definition of the exponential integral is

\[
E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt, \quad x > 0, \quad n = 0, 1, \ldots \quad (6.3.1)
\]

The function defined by the principal value of the integral

\[
\text{Ei}(x) = -\int_{-\infty}^x \frac{e^{-t}}{t} dt = \int_x^\infty \frac{e^t}{t} dt, \quad x > 0 \quad (6.3.2)
\]

is also called an exponential integral. Note that \( \text{Ei}(-x) \) is related to \( -E_1(x) \) by analytic continuation.

The function \( E_n(x) \) is a special case of the incomplete gamma function

\[
E_n(x) = x^{n-1} \Gamma(1 - n, x) \quad (6.3.3)
\]