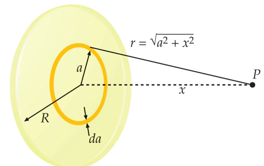
PHY204 Lecture 10 [rln10]

Electric Potential of Charged Disk

- Area of ring: $2\pi a da$
- Charge on ring: $dq = \sigma(2\pi a da)$
- Charge on disk: $Q = \sigma(\pi R^2)$



Find the electric potential at point P on the axis of the disk.

- $dV = k\dfrac{dq}{\sqrt{x^2+a^2}} = 2\pi\sigma k\dfrac{ada}{\sqrt{x^2+a^2}}$
- $V(x) = 2\pi\sigma k\displaystyle\int_0^R \frac{ada}{\sqrt{x^2+a^2}} = 2\pi\sigma k\left[\sqrt{x^2+a^2}\right]_0^R = 2\pi\sigma k\left[\sqrt{x^2+R^2} - |x|\right]$

Electric potential at large distances from the disk ($|x| \gg R$):

$$V(x) = 2\pi\sigma k|x|\left[\sqrt{1+\frac{R^2}{x^2}} - 1\right] \simeq 2\pi\sigma k|x|\left[1+\frac{R^2}{2x^2} - 1\right] = \frac{k\sigma\pi R^2}{|x|} = \frac{kQ}{|x|}$$

tsl82

We pick up the thread from the previous lecture with the electric potential generated by another charged object: a uniformly charged disk of radius R.

We assemble the disk from concentric rings for which we have earlier calculated the potential at points on the axis (here the x-axis). The result dV for a ring of radius a and width da is an adaptation of the previous result.

What remains to be done is summing up the contributions to the potential of rings with radii across the range $0 \leq a \leq R$. This amounts to an integral as carried out on the slide.

The result is the function $V(x)$ representing the electric potential for points on the axis a distance x away from the disk.

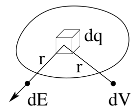
If we had picked a point away from the x-axis, at a point x, y, z in some coordinate system (not shown) we would need to calculate (with considerable additional effort) a function $V(x, y, z)$.

For large distances, $|x| \gg R$, we can expand the expression for $V(x)$ as shown in the last line on the slide and recover the electric potential of a point charge. From afar, a charged object of any shape acts like a point charge.

Electric Field and Electric Potential

Determine the field or the potential from the source (charge distribution):

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r^2}\hat{r}$$



$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}$$

Determine the field from the potential: $\vec{E} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}$

Determine the potential from the field: $V = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{s}$

- Systems with $\vec{E} = E_x(x)\hat{i}$: $E_x = -\frac{dV}{dx} \quad \Leftrightarrow \quad V(x) = -\int_{x_0}^{x} E_x dx$
- Application to charged ring: $E_x = \frac{kQx}{(x^2+a^2)^{3/2}} \Leftrightarrow V = \frac{kQ}{\sqrt{x^2+a^2}}$
- Application to charged disk (at $x > 0$): $E_x = 2\pi\sigma k\left[1 - \frac{x}{\sqrt{x^2+R^2}}\right] \Leftrightarrow V = 2\pi\sigma k\left[\sqrt{x^2+R^2} - x\right]$

tsl85

We now take a closer look at the relation between electric potential V and electric field $\vec{E}$. Previously we have calculated either quantity independently as an integral over the distribution of charge on objects of specific shape.

While field and potential can be determined independently as different integrals over charge distributions, the results are not independent.

If we know the electric potential $V(x, y, z)$ generated by a charge configuration, we can derive from that function three functions $E_x(x, y, z)$, $E_y(x, y, z)$, and $E_z(x, y, z)$, which represent the components of the electric field.

Conversely, if we know the electric field as those three functions, we can derive from them the electric potential as an integral of the dot product $\vec{E}\cdot d\vec{s}$ from some reference point $\vec{r}_0$ to the point $\vec{r} = (x, y, z)$.

How do we differentiate the function $V(x, y, z)$ in three different ways? How do we integrate $\vec{E}\cdot d\vec{s}$ along a specific path between two points? These are questions that we are going to answer in a gentle progression from simple to more complex scenarios.

If in a particular case the potential depends on one coordinate only, $V = V(x)$, then the associated electric field only has one component, $\vec{E} = E_x(x)\hat{i}$. The function $E_x(x)$ can be calculated from the function $V(x)$ via derivative.

Conversely, the potential $V(x)$ at position x can be calculated from the function $E_x(x)$ via an integral. Both operation are spelled out in the first item on the slide and then applied, in the second and third items, to results previously established for the cases of a charged ring and a charged disk.

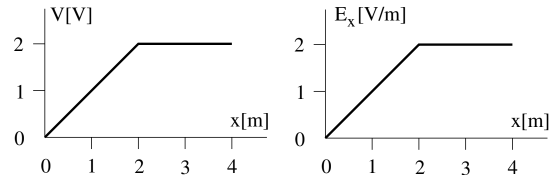
As a caveat we note that the relation between electric field and electric potential discussed here is only exact in electrostatics. There are additional sources of $\vec{E}$, not derivable from V, in electrodynamics (a later topic).

For given electric potential $V(x)$find the electric field

(a) $E_x(1\text{m})$,
(b) $E_x(3\text{m})$.

For given electric field $E_x(x)$ and given reference potential potential $V(0) = 0$ find the electric potential

(c) $V(2\text{m})$,
(d) $V(4\text{m})$.



tsl88

On this page and the next, we practice the calculation of electric field from electric potential and vice versa for the case where only one spatial coordinate is in play. We are dealing with the scalar quantity $V = V(x)$ and the vector quantity $\vec{E} = E_x(x)\hat{\imath}$. The operations to be performed are

$$E_x = -\frac{dV}{dx}, \quad V(x) = -\int_{x_0}^{x} E_x dx.$$

In the graph on the left, the function $V(x)$ is given graphically and we wish to know the electric field at two positions. The derivative dV/dx is determined by the slope of the curve at those two positions. It is positive at the first point and zero at the second. Taking into account the minus sign in the relation between potential and field we thus obtain the results,

$$E_x(1\text{m}) = -1\text{V/m}, \quad E_x(3\text{m}) = 0.$$

In the graph on the right, it is the function $E_x(x)$ that is given graphically, from which we wish to determine the electric potential at two positions. For that purpose we have to choose a reference position and perform the integral shown above. The most convenient reference point in this case is $x_0 = 0$. The integral is the area under the curve. For the position $x = 2\text{m}$, the area is that of a triangle. For the position $x = 4\text{m}$ we have to add the area of a square to that of the triangle. Then we must not forget the minus sign in the relation between field and potential. The results are

$$V(2\text{m}) = -2\text{V}, \quad V(4\text{m}) = -6\text{V}.$$

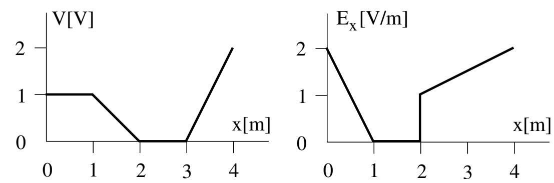
Electric Potential and Electric Field in One Dimension (2)

For given electric potential $V(x)$ find the electric field

(a) $E_x(0.5\text{m})$, (b) $E_x(1.5\text{m})$,
(c) $E_x(2.5\text{m})$, (d) $E_x(3.5\text{m})$.

For given electric field $E_x(x)$ and given reference potential potential $V(0) = 0$ find the electric potential

(e) $V(1\text{m})$, (f) $V(2\text{m})$, (g) $V(4\text{m})$.



tsl89

Here we can practice the skills we have learned on the previous page. The graphs representing the functions $V(x)$ and $E_x(x)$ are a bit more complex. When we choose $x_0 = 0$ as the reference position, it means that the electric potential vanishes at that point.

What matters in the curve on the left is the slope at four different positions. What matters in the curve on the right is the area under it between $x_0 = 0$ and two different positions to the right.

Here again are the relevant relations expressed analytically,

$$E_x = -\frac{dV}{dx}, \quad V(x) = -\int_{x_0}^{x} E_x dx.$$

When we carry out the operations we arrive at the following results:

$$E_x(0.5\text{m}) = 0, \quad E_x(1.5\text{m}) = 1\text{V/m}, \quad E_x(2.5\text{m}) = 0, \quad E_x(3.5\text{m}) = -2\text{V/m}.$$

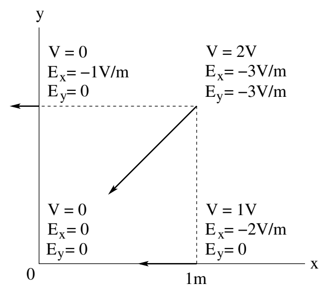
$$V(1\text{m}) = -1\text{V}, \quad V(2\text{m}) = -1\text{V}, \quad V(4\text{m}) = -4\text{V}.$$

If we know the electric potential at just one position, we cannot predict what the electric field is at the same position. We cannot take derivatives of numbers, only of functions. The electric field at a point does not depend on the value of the potential at that point, but on how the potential varies with position.

Electric Field from Electric Potential in Two Dimensions

- Given is the electric potential: $V(x,y) = ax^2 + bxy^3$ with $a = 1\text{V/m}^2$, $b = 1\text{V/m}^4$.
- Find the electric field: $\vec{E}(x,y) = E_x(x,y)\hat{i} + E_y(x,y)\hat{j}$ via partial derivatives.

$$E_x = -\frac{\partial V}{\partial x} = -2ax - by^3, \qquad E_y = -\frac{\partial V}{\partial y} = -3bxy^2$$



tsl86

If the electric potential varies in more than one direction, the associated electric field varies in as many directions, both in magnitude and direction.

Here we consider a situation where the electric potential is a function $V(x,y)$, explicitly stated on the slide. The electric field then has two components, which are both functions of two coordinates and can be determined via partial derivatives as carried out on the slide.

Partial derivatives are operations acted on functions with more than one variable such as $V(x,y)$. Unlike in an ordinary derivative, for which we can use the familiar notation,

$$V(x), \quad V'(x) = \frac{dV}{dx}, \quad V''(x) = \frac{d^2V}{dx^2}, \quad \ldots,$$

in partial derivatives our notation must be more explicit to avoid ambiguity:

$$V(x,y), \quad \frac{\partial V}{\partial x}, \quad \frac{\partial V}{\partial y}, \quad \frac{\partial^2 V}{\partial x^2}, \quad \frac{\partial^2 V}{\partial y^2}, \quad \frac{\partial^2 V}{\partial x \partial y}, \quad \ldots$$

There are two different first derivatives and three different second derivatives. In a partial derivative, we treat all variables as constants except the one with respect to which we perform the derivative. Note the special symbol in use.

Once the functions $E_x(x,y)$ and $E_y(x,y)$ are calculated from the function $V(x,y)$ via partial derivatives, we can determine the magnitude and the direction of the electric field at any point. We have done it before may times.

Electric Potential from Electric Field in Two Dimensions

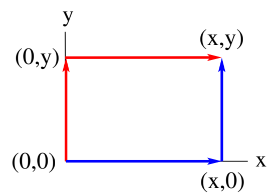
- Given is the electric field: $\vec{E} = -(2ax + by^3)\hat{i} - 3bxy^2\hat{j}$ with $a = 1\text{V/m}^2$, $b = 1\text{V/m}^4$.
- Find the electric potential $V(x,y)$ via integral along a specific path:

Red path $(0,0) \to (0,y) \to (x,y)$:

$$
\begin{aligned}
V(x,y) &= -\int_0^y E_y(0,y)dy - \int_0^x E_x(x,y)dx \\
&= 0 + \int_0^x (2ax + by^3)dx = ax^2 + bxy^3
\end{aligned}
$$

Blue path $(0,0) \to (x,0) \to (x,y)$:

$$
\begin{aligned}
V(x,y) &= -\int_0^x E_x(x,0)dx - \int_0^y E_y(x,y)dy \\
&= \int_0^x (2ax)dx + \int_0^y (3bxy^2)dy = ax^2 + bxy^3
\end{aligned}
$$



tsl87

If we know the electric field as a vector function $\vec{E}(x,y)$, such as the one calculated on the previous page, we can recover the electric potential $V(x,y)$ from it by performing an integral from a reference point, say $(0,0)$, to a generic point (x,y).

Unlike in the case of single coordinate, where there is just one direct path between two point, here we must choose one. On the slide we perform the integral along the red path and then again along the blue path.

When we integrate along the vertical stretch, we must keep the coordinate x fixed at the value that corresponds to the location of the vertical line. Those locations are different for the red and the blue vertical lines.

Likewise, when we integrate along a horizontal stretch, we must keep y fixed at the value corresponding to the location of the horizontal line.

There are infinitely may different paths between the points $(0,0)$ and (x,y). The complete integral is path-independent. Note, however, that the integral along the blue and red horizontal portions or the blue and red vertical portions are different.

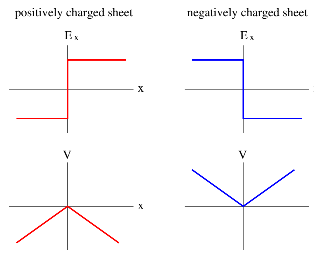
Be aware that the integral from $(0,0)$ to (x,y) is not path-independent for any vector function. Electrostatic fields are special. They can always be derived from an electric potential $V(x,y)$. The mathematical make-up of $\vec{E}(x,y)$ is such that the potential can be recovered without ambiguity.

Electric Potential of a Charged Plane Sheet

Consider an infinite plane sheet perpendicular to the x-axis at $x = 0$.

The sheet is uniformly charged with charge per unit area σ.

- Electric field (magnitude): $E = 2\pi k|\sigma| = \dfrac{|\sigma|}{2\epsilon_0}$
- Direction: away from (toward) the sheet if $\sigma > 0$ ($\sigma < 0$).
- Electric field (x-component): $E_x = \pm 2\pi k\sigma$.
- Electric potential: $V = -\int_0^x E_x dx = \mp 2\pi k\sigma x$.
- Here we have used $x_0 = 0$ as the reference coordinate.



tsl92

We complete this lecture by calculating the electric potential of several charged objects from the electric field determined earlier via Gauss's law. This involves an integration along a specific path. In all examples that we consider, only one coordinate is relevant, which makes the choice obvious.

We know from lecture 5 that the electric field on both sides of a uniformly charged plane sheet is uniform. It has the same direction and magnitude at any point in the space on one or the other side. The field is away from a positively charged sheet and toward a negatively charged sheet.

Since $\vec{E}$ is directed perpendicular to the plane of the sheet, we choose a path in that direction and declare it to be the x-axis with the sheet at $x = 0$. The position $x_0 = 0$ is a convenient choice of reference point. The integration is carried out on the slide.

Let us have a look at the graphical representations of field and potential. It is important that we can properly read such graphs.

The diagrams on top represent the x-component of the vector $\vec{E}$. Positive (negative) E_x means that the $\vec{E}$ is directed right (left). A positively charged particle is repelled from the positively charged sheet and attracted toward the negatively charged sheet by the force $\vec{F} = q\vec{E}$. The opposite is the case for a negatively charged particle.

The two diagrams below represent the scalar quantity V. The potential is zero at $x = 0$ in both cases. When we place a positively charged particle into a positive (negative) potential, it has positive (negative) potential energy. The opposite is the case if we place a negatively charged particle.

A charged particle is accelerated in the direction that leads to lower potential energy, which is consistent with the direction of force it experiences.

Electric Potential of a Uniformly Charged Spherical Shell

- Electric charge on shell: $Q = \sigma A = 4\pi\sigma R^2$

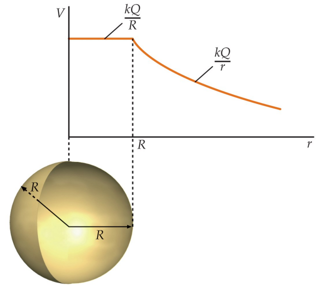
- Electric field at $r > R$: $E = \frac{kQ}{r^2}$
- Electric field at $r < R$: $E = 0$
- Electric potential at $r > R$:

$$V = -\int_{\infty}^{r} \frac{kQ}{r^2}\, dr = \frac{kQ}{r}$$

- Electric potential at $r < R$:

$$V = -\int_{\infty}^{R} \frac{kQ}{r^2}\, dr - \int_{R}^{r} (0) dr = \frac{kQ}{R}$$

- Here we used $r_0 = \infty$ as the reference value of the radial coordinate.



tsl93

We now determine the electric potential inside and outside a uniformly charged spherical shell. From an application of Gauss's law in lecture 6, we know that the electric field vanishes inside the shell. Outside the shell we have the familiar Coulomb field. The results are restated in the first three items.

The electric field has a radial direction, which makes the distance r from the center the only relevant coordinate in the integral expression of the electric potential. It is convenient the set the reference position at $r_0 = \infty$, where the electric field vanishes asymptotically.

The integral for a point on the outside is worked out in the fourth item and for a point on the inside in the fifth item. In the latter case, we split the range of integration into two intervals.

In the interval from ∞ to R, the Coulomb field applies Then in the interval from R to the final destination r, the field vanishes. The vanishing second integral implies that the electric potential is the same everywhere inside the shell.

We can recover the electric field from the electric potential by performing the derivative,

$$E = -\frac{dV}{dr},$$

Electric Potential of a Uniformly Charged Solid Sphere

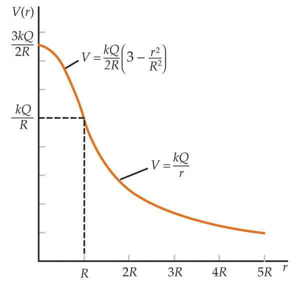
- Electric charge on sphere: $Q = \rho V = \frac{4\pi}{3}\rho R^3$
- Electric field at $r > R$: $E = \frac{kQ}{r^2}$
- Electric field at $r < R$: $E = \frac{kQ}{R^3} r$
- Electric potential at $r > R$:

$$V = -\int_{\infty}^{r} \frac{kQ}{r^2}\, dr = \frac{kQ}{r}$$

- Electric potential at $r < R$:

$$V = -\int_{\infty}^{R} \frac{kQ}{r^2}\, dr - \int_{R}^{r} \frac{kQ}{R^3}\, r\, dr$$

$$\Rightarrow V = \frac{kQ}{R} - \frac{kQ}{2R^3}\left(r^2 - R^2\right) = \frac{kQ}{2R}\left(3 - \frac{r^2}{R^2}\right)$$



tsl94

Here we repeat the same calculation for a uniformly charged solid sphere. We again start from the radial dependence of the electric field determined in lecture 6 via Gauss's law.

The result for the electric field on the outside only depends on the total charge. It does not matter whether that charge is in a point, on a shell, or uniformly distributed across the volume of a solid sphere.

The electric field inside the solid sphere does not vanish, which implies that the electric potential, inferred via integration, is not a constant.

The result is shown graphically on the slide. We see that the potential has a smooth maximum at the center of the sphere. A smooth maximum has zero slope, meaning zero derivative, which confirms that the electric field vanishes at the center of the sphere.

The graph for $V(r)$ has an inflection point at $r = R$. Here the slope has a maximum, which confirms that the electric field has maximum strength at the surface of the charged sphere.

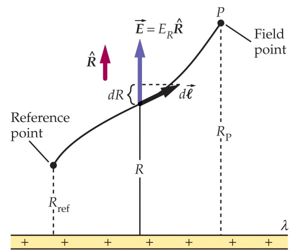
Electric Potential of a Uniformly Charged Wire

- Consider a uniformly charged wire of infinite length.
- Charge per unit length on wire: λ (here assumed positive).
- Electric field at radius r: $E = \dfrac{2k\lambda}{r}$.
- Electric potential at radius r:

$$V = -2k\lambda \int_{r_0}^{r} \frac{1}{r}\,dr = -2k\lambda\,[\ln r - \ln r_0]$$

$$\Rightarrow V = 2k\lambda \ln \frac{r_0}{r}$$

- Here we have used a finite, nonzero reference radius $r_0 \neq 0, \infty$.
- The illustration from the textbook uses R_{ref} for the reference radius, R for the integration variable, and R_p for the radial position of the field point.



tsl95

We have determined the electric field of a long charged rod or wire in lecture 5 using Gauss's law. The result, restated on the slide, holds at any point outside the wire.

The integral that produces the potential from the field is readily performed for an arbitrary reference value r_0. Unlike in the previous cases, neither $r_0 = 0$ nor $r_0 = \infty$ would be a suitable choice because the function,

$$V(r) = 2k\lambda \ln \frac{r_0}{r},$$

diverges for either choice, either to $-\infty$ or to $+\infty$. The most natural choice for r_0 is the radius of the wire.

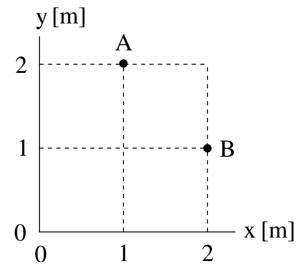
When we recover the electric field from this expression for the potential via the derivative, $E = -dV/dr$, the choice of r_0 makes no difference.

Electric Potential and Electric Field in Two Dimensions

Given is the electric potential $V(x,y) = cxy^2$ with $c = 1\text{V/m}^3$.

(a) Find the value (in SI units) of the electric potential V at point A.

(b) Find the components E_x, E_y (in SI units) of the electric field at point B.



tsl91

This is the quiz for lecture 10.

The instructions for part (b) are on page 5.