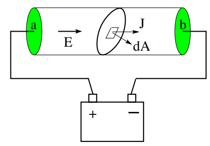
PHY204 Lecture 16 [rln16]

Resistor: Material and Device Perspectives

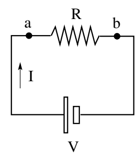
Material

- $\vec{E}$: electric field
- $\vec{J}$: current density
- ρ: resistivity
- $\sigma = 1/\rho$: conductivity
- $\vec{E} = \rho\vec{J}, \quad \vec{J} = \sigma\vec{E}$



Device

- V: voltage
- I: electric current
- R: resistance
- $V = RI$



Current from current density: $I = \int \vec{J} \cdot d\vec{A}$

Voltage from electric field: $V = -\int_a^b \vec{E} \cdot d\vec{s}$

tsl135

This first slide illustrates the bridge between the material level of description and the device level of description of a steady current driven by a power source.

What is a chunk of conducting materials connected to a real car battery on the material level becomes a circuit with two devices: a resistor and a voltage source.

The first itemized list names material attributes and states relations between them. We have discussed these quantities and relations at length in the previous lecture.

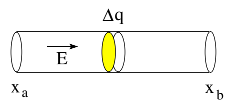
The second itemized list names device attributes and states one relation between them (Ohm's law). We shall use, for the most part, these quantities in the analysis of resistor circuits.

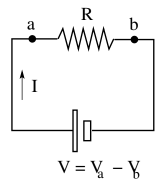
The two relations at the bottom of the slide relate device properties on the left to material properties on the right of each equation.

Power Dissipation in Resistor

Consider a resistor in the form of a uniform wire.

- Voltage between ends: $V \equiv V_a - V_b = E(x_b - x_a)$
- Displaced charge: $\Delta q = I\,\Delta t$





- Work done by electric field $\vec{E}$ on displaced charge Δq:

$$W_E = F(x_b - x_a) = E\,\Delta q\,(x_b - x_a) = V\Delta q = VI\Delta t$$

- Power dissipated in resistor: $P = \dfrac{W_E}{\Delta t} = VI = I^2R = \dfrac{V^2}{R}$
- SI unit: 1V·1A = (1J/C)·(1C/s) = 1J/s = 1W (Watt)

tsl140

In order to understand the power dissipation in a resistor when a voltage source drives a current through it, it helps to take another peek at what happens at the material level.

The electric field E inside the conductor pushes an amount of charge Δq a distance $x_a - x_b$ from left to right by exerting the force $F = \Delta q E$.

This involves work. The amount of work is expressed in four different ways in the third item. Note the shift from material-level attributes toward device-level attributes.

Power is the time rate of work, which is expressed in the fourth item in three different ways using device-level attributes only.

The relevant SI units are related in the last item.

Resistor Problem (1)

A heating element is made of a wire with a cross-sectional area $A = 2.60 \times 10^{-6}\text{m}^2$ and a resistivity $\rho = 5.00 \times 10^{-7}\Omega\text{m}$.

(a) If the element dissipates 5000W when operating at a voltage $V_1 = 75.0$V, what is its length L_1, its resistance R_1, and the current I_1 running through it?

(b) What must be the voltage V_2, the resistance R_2, and the length L_2 of a heating element made of the same wire if the same power should be generated with half the current?

tsl141

This exercise with focus on power dissipation in a resistor still straddles the bridge between material-level attributes and device-level attributes.

(a) The sequence of answering the questions matters.

$$I_1 = \frac{P}{V_1} = \frac{5000\text{W}}{75\text{V}} = 66.7\text{A}, \quad R_1 = \frac{V_1}{I_1} = \frac{75\text{V}}{66.7\text{A}} = 1.125\Omega.$$

$$L_1 = \frac{R_1 A}{\rho} = \frac{(1.125\Omega)(2.6 \times 10^{-6}\text{m}^2)}{5 \times 10^{-7}\Omega\text{m}} = 5.85\text{m}.$$

(b) Here again there is a logical sequence for the answers.

$$I_2 = \frac{1}{2}I_1 = 33.3\text{A}, \quad V_2 = \frac{P}{I_1} = 2V_1 = 150\text{V}.$$

$$R_2 = \frac{V_2}{I_2} = 4R_1 = 4.5\Omega, \quad L_2 = \frac{R_2 A}{\rho} = 4L_1 = 23.4\text{m}.$$

Resistor Problem (2)

A 1250W radiant heater is constructed to operate at 115V.

(a) What will be the current in the heater?
(b) What is the resistance of the heating coil?
(c) How much thermal energy is generated in one hour by the heater?

tsl142

One common use of resistors is electric heating. This simple exercise calculates what utility companies measure and quote when billing clients.

(a) Heating elements come with a label that tells us how much electric power they convert into heat.

$$I = \frac{P}{V} = \frac{1250\text{W}}{115\text{V}} = 10.9\text{A}.$$

(b) The "wattage" of a heating element that operates at a given voltage is dictated by the resistance of the heating coil.

$$R = \frac{V}{I} = \frac{115\text{V}}{10.9\text{A}} = 10.6\Omega.$$

(c) The energy unit quoted on electric bills is not the SI unit Joule.

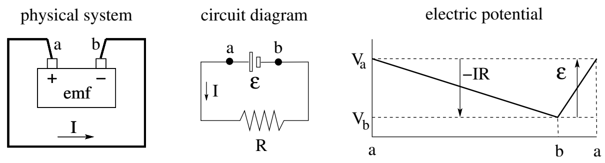
$$P\Delta t = (1250\text{W})(3600\text{s}) = 4.5 \times 10^6\text{J} = (1.25\text{kW})(1\text{h}) = 1.25\text{kWh}.$$

Our electric grid uses alternating current (ac) whereas this calculation has been carried out for direct current (dc). The results still hold true for ac if interpreted correctly (wait for lecture 34).

Direct Current Circuit

Consider a wire with resistance $R = \rho\ell/A$ connected to a battery.

- **Resistor rule**: In the direction of I across a resistor with resistance R, the electric potential drops: $\Delta V = -IR$.
- **EMF rule**: From the $(-)$ terminal to the $(+)$ terminal in an ideal source of emf, the potential rises: $\Delta V = \mathcal{E}$.
- **Loop rule**: The algebraic sum of the changes in potential encountered in a complete traversal of any loop in a circuit must be zero: $\sum \Delta V_i = 0$.



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In the remainder of this lecture, we focus on one-loop resistor circuits with steady currents. The simplest realization is a wire with resistance R connected to a power source generating a voltage $\mathcal{E}$. The acronym EMF stands for *electromotive force*.

In a one-loop circuit there is exactly one current. It is everywhere the same. At all points on the loop there is an electric potential. Here the current I has been declared to be counterclockwise (ccw). The analysis shows that it is positive.

All changes in potential happen across devices. Whether it is a step up or down depends on different criteria depending on the nature of the device.

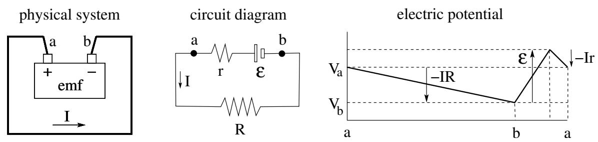
Resistor rule: What matters in the case of a resistor is the declared current direction. If we cross it in current direction, we go down in potential, $\Delta V = -IR$. If we cross it against the declared current direction, we go up in potential, $\Delta V = +IR$.

EMF rule: What matters in the case of a source of EMF (battery) is that the (+)-terminal is at a higher potential than the (−)-terminal, irrespective of the current direction. If we cross from the (+)-terminal to the (−)-terminal, we go down in potential, $\Delta V = -\mathcal{E}$. If we cross from the (−)-terminal to the (+)-terminal, we go up in potential, $\Delta V = +\mathcal{E}$.

Loop rule: It is our choice to loop around in the declared current direction or against it. What matters is that we take into account whether we go with the current or against the current across resistors.

Battery with Internal Resistance

- Real batteries have an internal resistance r.
- The terminal voltage $V_{ba} \equiv V_a - V_b$ is smaller than the emf $\mathcal{E}$ written on the label if a current flows through the battery.
- Usage of the battery increases its internal resistance.
- Current from loop rule: $\mathcal{E} - Ir - IR = 0 \quad \Rightarrow I = \dfrac{\mathcal{E}}{R+r}$
- Current from terminal voltage: $V_{ba} = \mathcal{E} - Ir = IR \quad \Rightarrow I = \dfrac{V_{ba}}{R}$



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A more realistic elementary one-loop circuit takes into account that batteries have an internal resistance. There are now two devices between the terminals of the physical battery: the source of EMF $\mathcal{E}$ and the internal resistor with resistance r.

The loop rule is implemented in the fourth item on the slide. We have declared the current to be ccw. We start at point b and go in current direction, first up across the EMF, then down across resistors r and R, before we return to point b.

The only unknown in the loop-rule equation is the current I. It turns out to be positive. Once we know the current I, we can determine the terminal voltage $V_{ba} \doteq V_a - V_b$ across the physical battery. It is smaller than the EMF $\mathcal{E}$ stated on its label.

A 3V-battery generates an EMF $\mathcal{E} = 3\text{V}$. It keeps doing that with usage. However, the internal resistance increases slowly as it is being used. In consequence, the terminal voltage is gradually diminished with usage. When used in a flashlight, the light gets dimmer as the internal resistance in the battery builds up.

Resistor Circuit (4)

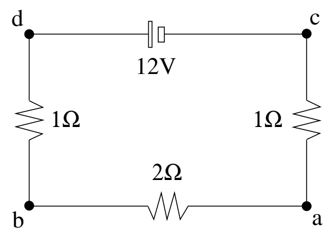
Consider the resistor circuit shown.

(a) Find the direction of the positive current (cw/ccw).

(b) Find the magnitude of the current.

(c) Find the voltage $V_{ab} = V_b - V_a$.

(d) Find the voltage $V_{cd} = V_d - V_c$.



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This is a one-loop circuit with a steady current. The current direction is a matter of choice. The direction of positive current is a result.

To begin the analysis, we must make two choices: (i) Do we declare the current to be clockwise (cw) or counterclockwise (ccw)? (ii) In the implementation of the loop rule, where do we start the loop and do we go around cw or ccw? We now analyze the circuit with two different sets of choices.

First set of choices: I is ccw; we go cw around the loop starting at point c.

$$I(1\Omega) + I(2\Omega) + I(1\Omega) - 12\text{V} = 0 \quad \Rightarrow \; I = +3\text{A}.$$

$$\Rightarrow \; V_{ab} = (+3\text{A})(2\Omega) = +6\text{V} \text{ or } V_{ab} = -(3\text{A})(1\Omega) + 12\text{V} - (3\text{A})(1\Omega) = +6\text{V}.$$

$$V_{cd} = +12\text{V} \text{ or } V_{cd} = (3\text{A})(1\Omega) + (3\text{A})(2\Omega) + (3\text{A})(1\Omega) = +12\text{V}.$$

Second set of choices: I is cw; we go cw around the loop starting at point c.

$$-I(1\Omega) - I(2\Omega) - I(1\Omega) - 12\text{V} = 0 \quad \Rightarrow \; I = -3\text{A}.$$

$$\Rightarrow \; V_{ab} - (-3\text{A})(2\Omega) = +6\text{V} \text{ or } V_{ab} = +(-3\text{A})(1\Omega) + 12\text{V} + (-3\text{A})(1\Omega) = +6\text{V}.$$

$$V_{cd} = +12\text{V} \text{ or } V_{cd} = -(-3\text{A})(1\Omega) - (-3\text{A})(2\Omega) - (-3\text{A})(1\Omega) = +12\text{V}.$$

Depending on our choice of current direction, I comes out positive or negative. The positive current direction is ccw, indpendent of our choice.

When calculating V_{ab} we move from point a to point b one way or the other, now using the result for the current. Likewise for V_{cd}. The results are independent of the declared current direction.

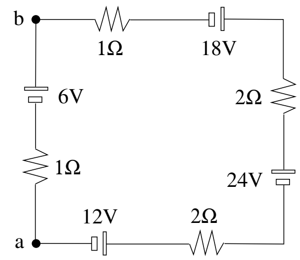
Resistor Circuit (6)

Consider the resistor circuit shown.

(a) Choose a current direction and use the loop rule to determine the current.

(b) Name the direction of positive current (cw/ccw).

(c) Find $V_{ab} \equiv V_b - V_a$ along two different paths.



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This is a one-loop circuit with four EMF sources and four resistors. Part (a) is explicit about the need to choose the current direction. Either choice will produce the correct result for parts (b) and (c).

Let us choose a ccw current and a loop that starts at point b and goes around cw, i.e. against the current.

Implementation of loop rule:

$$(1\Omega)I + 18\text{V} + (2\Omega)I - 24\text{V} + (2\Omega)I - 12\text{V} + (1\Omega)I + 6\text{V} = 0.$$

$$\Rightarrow \quad (6\Omega)I = 12\text{V} \quad \Rightarrow \quad I = 2\text{A}.$$

Potential change from point a to point b across the short way:

$$V_{ab} = (1\Omega)(2\text{A}) + 6\text{V} = 8\text{V}.$$

Potential change from point a to point b across the long way:

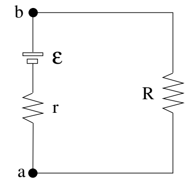
$$V_{ab} = 12\text{V} - (2\Omega)(2\text{A}) + 24\text{V} - (2\Omega)(2\text{A}) - 18\text{V} - (1\Omega)(2\text{A}) = 8\text{V}.$$

If we had chosen a cw current direction the result for the current would have been $I = -2\text{A}$ and the result for V_{ab} would have been the same.

Power in Resistor Circuit

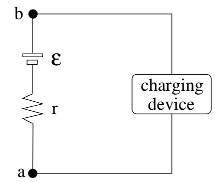
Battery in use

- Terminal voltage: $V_{ab} = \mathcal{E} - Ir = IR$
- Power output of battery: $P = V_{ab}I = \mathcal{E}I - I^2r$
  - Power generated in battery: $\mathcal{E}I$
  - Power dissipated in battery: I^2r
- Power transferred to load: $P = I^2R$



Battery being charged:

- Terminal voltage: $V_{ab} = \mathcal{E} + Ir$
- Power supplied by charging device: $P = V_{ab}I$
- Power input into battery: $P = \mathcal{E}I + I^2r$
  - Power stored in battery: $\mathcal{E}I$
  - Power dissipated in battery: I^2r



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Here our focus is on rechargeable batteries. Its positive terminal is at point b and its negative terminal at point a. Its internal resistance is r.

When the battery is in use, it is connected to a load, which is represented by by the external resistor with resistance R. A positive cw current I flows through the loop. We have calculated it earlier (on page 6).

The load experiences the terminal voltage V_{ab}, which is smaller than the EMF $\mathcal{E}$. The power output is the difference between the power generated and the power dissipated in the battery. The battery heats up while it is in use.

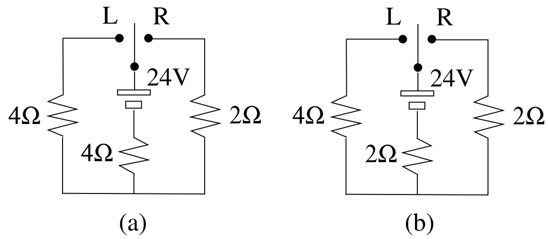
When the battery is being recharged, the charging device acts as a power source and drives a positive ccw current (or a negative cw current) through the battery.

During recharging, the terminal voltage V_{ab} is larger than the EMF $\mathcal{E}$. The power supplied by the charging device is, in part, stored in the battery for later use and, in part, dissipated in the internal resistance of the battery. The battery also heats up while it is being recharged.

Resistor Circuit (7)

Consider two 24V batteries with internal resistances (a) $r = 4\Omega$, (b) $r = 2\Omega$.

- Which setting of the switch (L/R) produces the larger power dissipation in the resistor on the side?



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This page is still about one-loop resistor circuits. When the switch is closed to left (L), the battery drives a positive ccw current through the load on the left. When the switch is closed to the right (R), the positive current is cw through the load on the right.

We are interested in the power $P = I^2R$ transferred from the battery to the load (external resistor with resistance R) in all four configurations. The calculations are straightforward.

(a) internal resistance $r = 4\Omega$

L: $I = 3\text{A}, \quad P = 36\text{W}$

R: $I = 4\text{A}, \quad P = 32\text{W}$

(b) internal resistance $r = 2\Omega$

L: $I = 4\text{A}, \quad P = 64\text{W}$

R: $I = 6\text{A}, \quad P = 72\text{W}$

The crucial point to be appreciated is that the power transfer in each circuit is higher when the internal and external resistances are the same.

This is the case when the switch has setting L in circuit (a) and setting R in circuit (b). The higher power transfer is not always associated with the larger current.

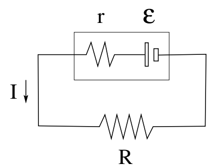
What we have discovered here is more general, as the next page will show.

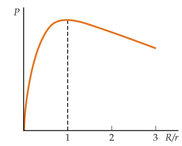
Impedance Matching

A battery providing an emf $\mathcal{E}$ with internal resistance r is connected to an external resistor of resistance R as shown.

For what value of R does the battery deliver the maximum power to the external resistor?

- Electric current: $\mathcal{E} - Ir - IR = 0 \quad \Rightarrow \; I = \dfrac{\mathcal{E}}{R+r}$
- Power delivered to external resistor: $P = I^2R = \dfrac{\mathcal{E}^2 R}{(R+r)^2} = \dfrac{\mathcal{E}^2}{r}\dfrac{R/r}{(R/r+1)^2}$
- Condition for maximum power: $\dfrac{dP}{dR} = 0 \quad \Rightarrow \; R = r$





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Here we keep the internal resistance r fixed and vary the external resistance R such that the power transfer has a maximum.

We begin with the loop rule to determine the current as we have done already several times. Then we write the power transfer as a function of R/r and plot the function.

We see that this function has a maximum. At its maximum, the function has zero slope, which means that its first derivative must be zero. When we take the derivative of the function and set it equal to zero, we find that the value of the external resistance must be equal to that of the internal resistance.

This is a special case of impedance matching. Impedance is a term used in ac circuits (to be discussed later), of which resistance is one realization. Impedance matching matters when pieces of electric equipment are assembled such as amplifiers and speakers.

Resistor Circuit (5)

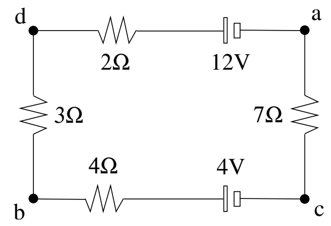
Consider the resistor circuit shown.

(a) Choose a current direction and use the loop rule to determine the current.

(b) Name the direction of positive current (cw/ccw).

(c) Find the potential difference $V_{ab} = V_b - V_a$.

(d) Find the voltage $V_{cd} = V_d - V_c$.



tsl152

This the quiz for lecture 16.

It is a variation of what we have done earlier.