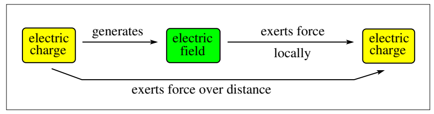
PHY204 Lecture 23 [rln23]

Electric Field of a Point Charge

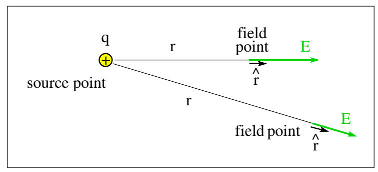


(1) Electric field $\vec{E}$ generated by point charge q: $\vec{E} = k\frac{q}{r^2}\hat{r}$

(2) Force $\vec{F}_1$ exerted by field $\vec{E}$ on point charge q_1: $\vec{F}_1 = q_1\vec{E}$

(1+2) Force $\vec{F}_1$ exerted by charge q on charge q_1: $\vec{F}_1 = k\frac{qq_1}{r^2}\hat{r}$ (static conditions)

- $\epsilon_0 = 8.854 \times 10^{-12}\text{C}^2\text{N}^{-1}\text{m}^{-2}$
- $k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9\text{Nm}^2\text{C}^{-2}$
- SI unit of E: [N/C]



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The focus of this page and the next is on charged particles as sources of electric and magnetic fields.

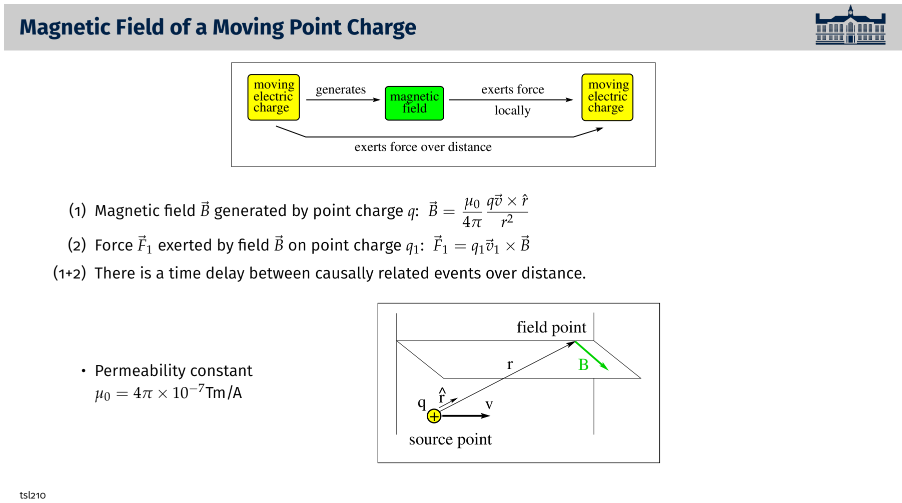
Here we review a slide from earlier in the course. A point charge q at rest is the source of a static (i.e. time-independent) electric field $\vec{E}$ everywhere in space. The field direction is radial, pointing away from a positive charge (as shown) and pointing toward a negative charge (not shown).

When a second point charge q_1 is present, it experiences an electric force, $\vec{F}_1 = q_1\vec{E}$, exerted by the electric field at its position. If the source of that field is the charge q, then the force can be reinterpreted as a force over distance between the two point charges.

If the source q is moving, then the expression for the electric field is more complicated. Expression (2) is still valid, but the field $\vec{E}$ in that expression is different from (1). Therefore, (1+2) is no longer valid.

The complication has to do with the fact that when q starts moving, the surrounding field does not move rigidly with its source. The news that the source got moving travels outward at the speed of light, producing a time-dependent distortion in the field.

A further complication that comes into play is that moving charges also generate a different field (see next page).



Consider a point charge q moving with constant velocity $\vec{v}$ as shown on the slide. It generates a magnetic field $\vec{B}$ with magnitude and direction as captured by expression (1) on the slide.

Unlike a static electric field, the magnetic field is not radial. Its direction is the result of a cross product involving the velocity $\vec{v}$ of the source and the distance vector $\vec{r}$ from source point to field point as factors. The product vector is perpendicular to the plane spanned by these two vectors.

In the graph, the two factor vectors are in the vertical plane. The cross product is perpendicular to that plane. The right-hand rule tells you that the field $\vec{B}$ is out of the plane. If q were negative, $\vec{B}$ at the same position would be into the plane. Recall that $\hat{r} \doteq \vec{r}/r$ is a unit vector in the direction of $\vec{r}$.

Take note of and get familiar with the *permeability constant* μ_0. Its role in magnetism is similar to the role of the *permittivity constant* ϵ_0 in electrostatics. In electrodynamics, their roles become entangled, for example, in the speed of light,

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 299792458...\text{m/s}.$$

Keep in mind that we are dealing with a dynamic situation. As the source moves on, the vector $\vec{r}$ changes unless we move the field point along. The usefulness of expression (1) is quite limited for that reason.

If you place a second point charge q_1 into a magnetic field $\vec{B}$, it experiences a magnetic force as expressed in (2). The validity of this expression is more general than (1). The two expressions can only be combined if time delays between cause and effect are accounted for.

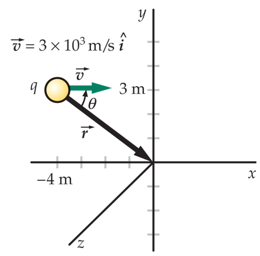
Magnetic Field Application (1)

A particle with charge $q = 4.5\text{nC}$ is moving with velocity $\vec{v} = 3 \times 10^3 \text{m/s}\hat{i}$.

Find the magnetic field generated at the origin of the coordinate system.

- Position of field point relative to particle: $\vec{r} = 4\text{m}\hat{i} - 3\text{m}\hat{j}$
- Distance between Particle and field point: $r = \sqrt{(4\text{m})^2 + (3\text{m})^2} = 5\text{m}$
- Magnetic field:

$$\begin{aligned} \vec{B} &= \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \vec{r}}{r^3} \\ &= \frac{\mu_0}{4\pi}\frac{q(3 \times 10^3 \text{m/s}\hat{i}) \times (4\text{m}\hat{i} - 3\text{m}\hat{j})}{(5\text{m})^3} \\ &= -\frac{\mu_0}{4\pi}\frac{q(3 \times 10^3 \text{m/s}\hat{i}) \times (3\text{m}\hat{j})}{(5\text{m})^3} \\ &= -3.24 \times 10^{-14}\text{T}\hat{k}. \end{aligned}$$



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Here we have a simple quantitative application of the source expression introduced on the previous page. The source point is the instantaneous position of the moving particle and the field point has been chosen to be at the origin of the coordinate system. The vector $\vec{r}$ marks the field point relative to the source point.

Note that the first expression for the magnetic field as taken from the previous page is rewritten in the second expression by substituting the definition, $\hat{r} \doteq \vec{r}/r$, of the unit vector. The second expression is often more practical.

Confirm, using the right-hand rule, that the direction of $\vec{B}$ is into the plane, just as the calculation, which uses components, predicts.

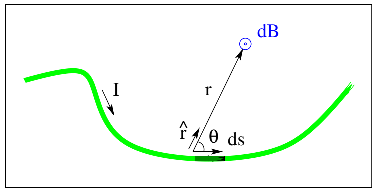
In preparation of what is waiting for us on the next page, we extract from the first expression for $\vec{B}$ an expression for its magnitude:

$$B = \frac{\mu_0}{4\pi}\frac{q|\vec{v}|\,|\hat{r}|\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{qv\sin\theta}{r^2}.$$

Law of Biot and Savart

- Current element: $Id\vec{s} = dq\vec{v}$ [1Am = 1Cm/s]
- Magnetic field of current element: $dB = \frac{\mu_0}{4\pi}\frac{dqv\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{Ids\sin\theta}{r^2}$
- Vector relation: $d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{s}\times\hat{r}}{r^2}$
- Magnetic field generated by current of arbitrary shape:

$$\vec{B} = \frac{\mu_0}{4\pi}\int\frac{Id\vec{s}\times\hat{r}}{r^2} \quad \text{(Law of Biot and Savart)}$$



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Whereas the motion of individual charged particles always produce magnetic fields that are necessarily time-dependent, it is possible to a produce static magnetic field by steady electric currents.

A steady current is produced by a steady stream of mobile charge carriers inside a conductor. While each charge carrier produces a time-dependent magnetic field at every field point, the current averages out this time-dependence into a static field.

The collective motion of charge carriers inside a short conductor segment can be expressed as the collective charge dq moving at drift velocity $\vec{v}$ or, alternatively, as a current I flowing in the directed segment $d\vec{s}$.

This equivalence allows us to transcribe the expression for the magnetic-field magnitude from the previous page to the one shown in the second line on the slide of this page.

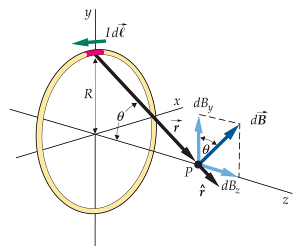
The associated vector expression is recovered in the third line. It expresses the static field generated by a segment $d\vec{s}$ of steady current I a distance r away from the field point.

Calculating the magnetic field $\vec{B}$ originating from a steady current of arbitrary shape at a field point of choice then amounts to a summation over infinitesimal segments $d\vec{s}$, converted into the integral as shown on the slide. This expression is known as the law of Biot and Savart.

In the following, we apply this expression for selected applications that are not too challenging mathematically.

Magnetic Field of Circular Current

- Law of Biot and Savart: $dB = \frac{\mu_0}{4\pi}\frac{Id\ell}{z^2+R^2}$
- $dB_z = dB\sin\theta = dB\frac{R}{\sqrt{z^2+R^2}}$
  $\Rightarrow\ dB_z = \frac{\mu_0 I}{4\pi}\frac{Rd\ell}{(z^2+R^2)^{3/2}}$
- $B_z = \frac{\mu_0 I}{4\pi}\frac{R}{(z^2+R^2)^{3/2}}\int_0^{2\pi R} d\ell$
  $\Rightarrow\ B_z = \frac{\mu_0 I}{2}\frac{R^2}{(z^2+R^2)^{3/2}}$
- Field at center of ring $(z=0)$: $B_z = \frac{\mu_0 I}{2R}$
- Magnetic moment: $\mu = I\pi R^2$
- Field at large distance $(z \gg R)$: $B_z \simeq \frac{\mu_0}{2\pi}\frac{\mu}{z^3}$



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Here we consider a current flowing around a circle placed in the xy-plane with the center at the origin of the coordinate system. For field points that lie on the z-axis the calculation is quite manageable.

A key simplification is achieved if we split the magnetic field $d\vec{B}$ generated by each segment into components dB_z parallel to the z-axis and a vector $d\vec{B}_\perp$ perpendicular to it. The latter (named dB_y on the slide) averages out to zero when we sum contribution from segments around the circle.

The integral of dB_z is very simple and performed on the slide. The magnetic field $\vec{B}$ is always pointing in the positive z-direction. When we say that, we must also say that the current flows in the direction shown. If I flows in the opposite direction, then $\vec{B}$ switches direction as well.

Note that the expression for the magnetic field simplifies a great deal if we choose the field point at the center of the circle $(z = 0)$.

Recall the magnetic dipole moment of a circular current loop. The direction of $\vec{\mu}$ for the case shown is in the positive z-direction. We have previously learned to use the right-hand rule for that determination.

For a magnetic dipole aligned with the z-axis and positioned at its origin, the magnetic field a large distance away on z-axis then only depends on the dipole moment as shown. A corresponding expression introduced earlier exists for electric dipole moments.

Magnetic Field Application (11)

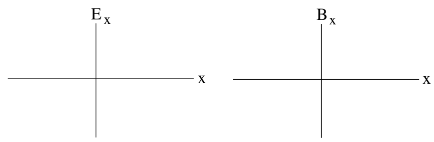
The electric field E_x along the axis of a charged ring and the magnetic field B_x along the axis of a circular current loop are

$$E_x = \frac{Q}{4\pi\epsilon_0}\frac{x}{(x^2+R^2)^{3/2}}, \qquad B_x = \frac{\mu_0 I}{2}\frac{R^2}{(x^2+R^2)^{3/2}}$$

(a) Simplify both expressions for $x = 0$.

(b) Simplify both expressions for $x \gg R$.

(c) Sketch graphs of $E_x(x)$ and $B_x(x)$.



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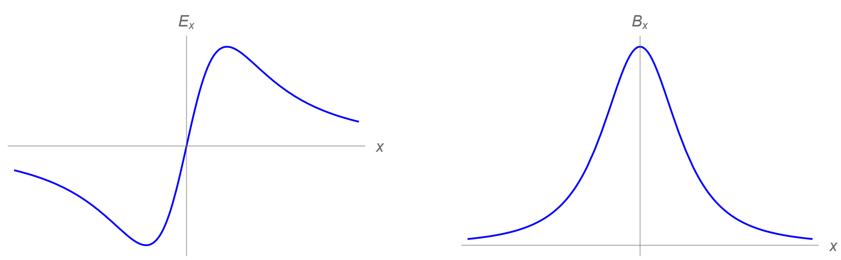
Consider two ring of radius R, each positioned such that the x-axis goes right through its center and is perpendicular to its plane.

The two expressions side by side represent (on the left) the electric field generated by a charged ring and (on the right) the magnetic field generated by a circular current, in both case for field point on the x-axis.

The two expressions look similar at first glance. One features the permeability constant, the other the permeability constant. However, understanding the differences is important.

When we graph the two functions, we see key differences at once.



The electric field changes direction from left to right, whereas the magnetic field always points to the right.

At the center of the ring, we have $E_x = 0, \quad B_x = \dfrac{\mu_0 I}{2R}.$

At large distances, $x \gg R$, the the electric field approaches zero more slowly than the magnetic field. Viewed from far, the charged ring is an electric monopole, whereas the current ring is a magnetic dipole.

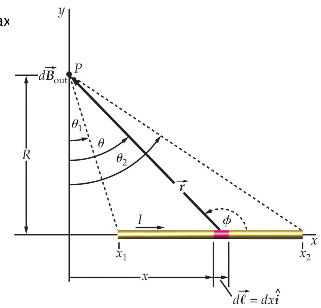
$$E_x \sim \frac{Q}{4\pi\epsilon_0 x^2}, \quad B_x \sim \frac{\mu_0 I R^2}{2x^3}.$$

Magnetic Field Generated by Current in Straight Wire (1)

Consider a field point P that is a distance R from the ax

- $dB = \frac{\mu_0}{4\pi}\frac{Idx}{r^2}\sin\phi = \frac{\mu_0}{4\pi}\frac{Idx}{r^2}\cos\theta$
- $x = R\tan\theta \;\Rightarrow\; \frac{dx}{d\theta} = \frac{R}{\cos^2\theta} = \frac{R}{R^2/r^2} = \frac{r^2}{R}$
- $dB = \frac{\mu_0}{4\pi}\frac{I}{r^2}\frac{r^2 d\theta}{R}\cos\theta = \frac{\mu_0}{4\pi}\frac{I}{R}\cos\theta d\theta$
- $B = \frac{\mu_0}{4\pi}\frac{I}{R}\int_{\theta_1}^{\theta_2}\cos\theta d\theta$

  $= \frac{\mu_0}{4\pi}\frac{I}{R}(\sin\theta_2 - \sin\theta_1)$
- Length of wire: $L = R(\tan\theta_2 - \tan\theta_1)$



Wire of infinite length: $\theta_1 = -90^\circ$, $\theta_2 = 90^\circ \;\Rightarrow\; B = \frac{\mu_0 I}{2\pi R}$

tsl216

The goal here is to construct a practical expression for the magnetic field generated by finite but not infinitesimal, straight and thin segment of steady current. Such segments do, of course, not exist in isolation, but many currents of practical relevance can be assembled out of such segments linked together.

The expression that comes out of the (somewhat lengthy) derivation sketched on the slide is shown on the fourth bullet line. It contains the two angles θ_1, θ_2 and the distance R.

If we wish to use this expression in an application, we must introduce a coordinate system such that the current segment lies along the x-axis and the field point lies on the y-axis. The distance between the field point and the x-axis is R. The line between the field point and the front (rear) end of current segment is at angle θ_2 (θ_1) from the y-axis.

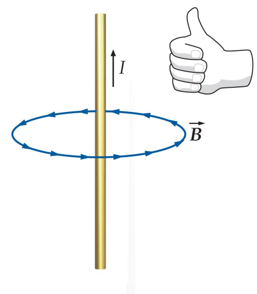
If, on the graph shown, we move the field point to the right, then the two angles gradually decrease and go negative, first θ_1 and then θ_2. The direction of $\vec{B}$ is $\odot$ when the current flows to the right as shown. Switching the current direction from $\rightarrow$ to $\leftarrow$ causes $\vec{B}$ to switch from $\odot$ to $\otimes$.

When we consider a situation where the field point is close to a wire, then both angles are close to right angles with opposite sign as indicated in the last line of the slide. The magnetic field is then well approximated by that of an infinitely long straight current. The expression for the magnitude of the current now has a very simple structure and depends only on the distance of the field point from the line of current.

Magnetic Field Generated by Current in Straight Wire (2)

Consider a current I in a straight wire of infinite length.

- The magnetic field lines are concentric circles in planes prependicular to the wire.
- The magnitude of the magnetic field at distance R from the center of the wire is $B = \frac{\mu_0 I}{2\pi R}$.
- The magnetic field strength is proportional to the current I and inversely proportional to the distance R from the center of the wire.
- The magnetic field vector is tangential to the circular field lines and directed according to the right-hand rule.



tsl217

Here we take a closer look at determining the direction of the magnetic field generated by a current in a wire of any shape.

We begin with a long straight wire carrying a current flowing ↑ as shown. The vector $\vec{B}$ must then be tangential to the circular line shown. The right-hand rule identifies the direction.

The circular line can is a magnetic-field lines. All magnetic field lines are closed (for reasons explained shortly).

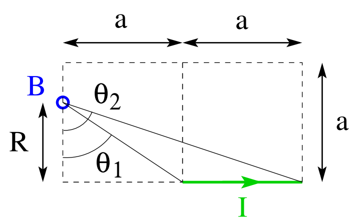
We can apply the right-hand rule to currents in wires of any shape. For example, if we wrap the fingers of our right hand around any part of the circle shown on page 5 with the thumb in current direction, we conclude that the magnetic field points to the right inside the circle and to the left outside the circle. This remains true for any point in the plane of the circle.

Magnetic Field Generated by Current in Straight Wire (3)

Consider the magnetic field $\vec{B}$ in the limit $R \to 0$.

- $B = \frac{\mu_0}{4\pi}\frac{I}{R}(\sin\theta_2 - \sin\theta_1)$
- $\sin\theta_1 = \frac{a}{\sqrt{a^2+R^2}} = \frac{1}{\sqrt{1+\frac{R^2}{a^2}}} \simeq 1 - \frac{1}{2}\frac{R^2}{a^2}$
- $\sin\theta_2 = \frac{2a}{\sqrt{4a^2+R^2}} = \frac{1}{\sqrt{1+\frac{R^2}{4a^2}}} \simeq 1 - \frac{1}{2}\frac{R^2}{4a^2}$
- $B \simeq \frac{\mu_0}{4\pi}\frac{I}{R}\left(1 - \frac{1}{2}\frac{R^2}{4a^2} - 1 + \frac{1}{2}\frac{R^2}{a^2}\right)$

  $= \frac{\mu_0 I}{4\pi}\frac{3R}{8a^2} \stackrel{R\to 0}{\longrightarrow} 0$



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Here we return to a straight segment of current of finite length and investigate what happens when we reduce the distance R of the field point from the line of current toward zero such that it approaches the line to the side of the current segment as shown on the slide.

In this case, both the numerator and the denominator of the expression in the first bullet line approach zero, which leaves us guessing whether the strength of $\vec{B}$ goes to infinity or to zero or whether it approaches a finite value.

A closer examination of this question requires that we expand the angles in powers of R/a as is done on the slide. Mathematically speaking, this is a binomial expansion. The answer is that the the field strength B approaches zero.

This result is important to know in many applications.

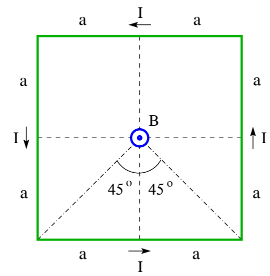
Note that if we made R smaller and smaller at a location halfway between the front and rear ends of the segment, then only the numerator approaches zero, which makes the field strength approach a very large value when the distance reaches the surface of a very thin wire.

What the magnetic field is inside the wire, will be a topic of investigation for which we use a different method of analysis (stay tuned).

Magnetic Field at Center of Square-Shaped Wire

Consider a current-carrying wire bent into the shape of a square with side $2a$.

Find direction and magnitude of the magnetic field generated at the center of the square.



$$B = 4\frac{\mu_0}{4\pi}\frac{I}{a}\Big[\sin(45^\circ) - \sin(-45^\circ)\Big] = \frac{\sqrt{2}\mu_0 I}{\pi a}.$$

tsl518

What is the magnetic field $\vec{B}$ at the center of a current-carrying wire shaped into a square of side $2a$?

Let us determine first the direction, which is $\odot$ as indicated. To confirm that, we apply the right-hand rule using any one of the four sides.

When we determine the strength (magnitude) of $\vec{B}$ we think of the square as four segments of equal length $2a$ linked together. Let us begin with the bottom side. The angles are $\theta_2 = 45^\circ$ and $\theta_1 = -45^\circ$ and the relevant distance to the field point is a. Then we realize that each segment contributes equally. The result is shown on the slide.

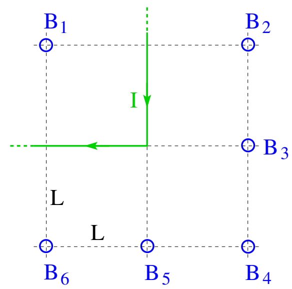
Magnetic Field Application (6)

A current-carrying wire is bent into two semi-infinite straight segments at right angles.

(a) Find the direction ($\odot, \otimes$) of the magnetic fields $B_1, \ldots, B_6$.

(b) Name the strongest and the weakest fields among them.

(c) Name all pairs of fields that have equal strength.



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A smart way to analyze this situation is to consider two types of current segments:

- A semi-infinite current with the field point a distance L away from the edge such B_5 from the horizontal green line. The field strength associated with this configuration is

$$B_s = \frac{\mu_0 I}{4\pi L}.$$

- A segment of length L such as the horizontal green line inside the big dashed square with the field point positioned as B_6 or B_5. The field strength associated with such a configuration is

$$B_f = \frac{\mu_0 I}{4\pi L}\frac{1}{\sqrt{2}}.$$

Next we declare that any field out of the page ($\odot$) is counted positively and any field pointing into the page ($\otimes$) negatively.

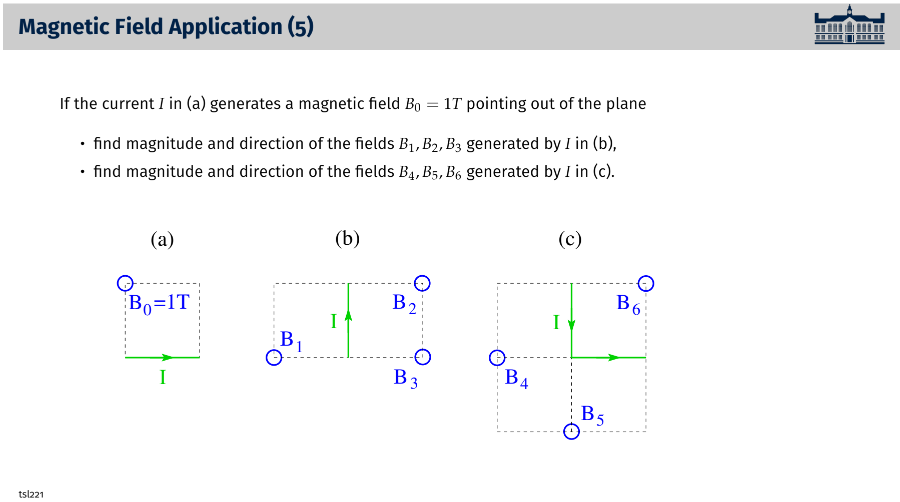
Now we assemble all current configurations viewed from the six field points from the two pieces analyzed above with the appropriate signs. The results should like as follows:

$$B_1 = -B_s\left[1 + \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}\right] \simeq -3.4B_s,$$

$$B_2 = B_6 = B_s\left[1 + \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}}\right] \simeq 1.4B_s,$$

$$B_3 = B_5 = B_s,$$

$$B_4 = B_s\left[1 - \frac{1}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}}\right] \simeq 0.6B_s.$$



This is the quiz for lecture 23.

Consider the current segment shown in configuration (a). We know that the field $\vec{B}_0$ is pointing $\odot$ and are told that its strength is 1T. The field at the upper right corner of the same configuration would be exactly the same for symmetry reason. This is readily verified by changing the angles in the general expression,

$$B = \frac{\mu_0}{4\pi}\frac{I}{R}\big(\sin\theta_2 - \sin\theta_1\big).$$

For the field point marked, the angles are $\theta_2 = 45°$, $\theta_1 = 0$ whereas for the field point at the upper right corner they are $\theta_2 = 0$, $\theta_1 = -45°$. Replacing both angles leaves the result invariant.

With this information in hand, find direction and magnitude of the magnetic field at point 1 though 6 in configurations (b) and (c). The possible answers for direction are $\odot$ and $\otimes$ and for magnitude an integer number in units of Tesla.