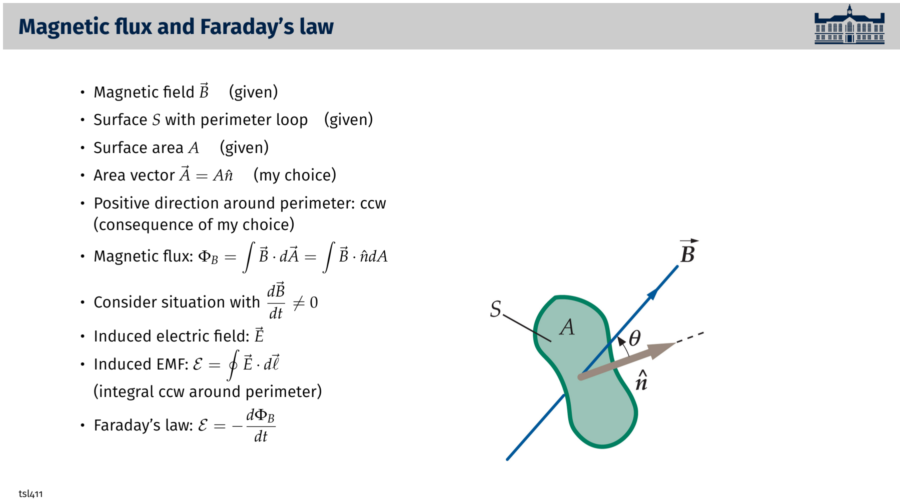
PHY204 Lecture 28 [rln28]



We begin this lecture be reviewing the essence of Faraday's law. On the next two pages we discuss Lenz's rule, a useful tool for predicting the direction of induced currents.

In the itemized list on the slide, the first three items are given quantities in this application. The next two items establish the convention adopted. The two are related. Either we specify the direction of $\vec{A}$ or the positive loop direction. The other follows by right-hand rule.

The choice of convention has no bearing on the time-varying magnetic field $\vec{B}(t)$ or the induced electric field $\vec{E}$.

The convention determines the sign ($\pm$) of the quantities Φ_B, $d\Phi_B/dt$, and $\mathcal{E}$, but the meaning of that sign depends on the convention. Adopting a convention is necessary to carry out the calculation.

Once we have determined the induced EMF and it comes out positive (negative), we recall the adopted convention, which associates positive (negative) with ccw (cw).

In this application, the cause of induction is a time-varying magnetic field, producing a time-varying magnetic flux through a rigid loop. The flux changes if either the direction of $\vec{B}$ or its magnitude changes.

In a different application, we might keep the magnetic field constant and change the orientation or the shape of the loop, which also induces an EMF.

Lenz's Rule (1)

The induced emf and induced current are in such a direction as to oppose the cause that produces them.

- Lenz's rule is a statement of negative feedback.
- The cause is a change in magnetic flux through some loop.
- The loop can be real or fictitious.
- What opposes the cause is a magnetic field generated by the induced emf.
  - If the loop is a conductor the opposing magnetic field is generated by the induced current as stated in the law of Biot and Savart or in the restricted version of Ampère's law.
  - If the loop is not a conductor the opposing magnetic field is generated by the induced electric field as stated by the extended version of Ampère's law (to be discussed later).

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Lenz's rule is not a law of nature. It is not a substitute for Faraday's law, nor is it an amendment to it. Its status is often overrated in introductory texts. Lenz's rule is a useful tool for the prediction of induced currents.

Induction triggers a feedback in the following sense. A time-varying magnetic field produces an induced EMF. The induced EMF produces an induced current. The induced current generates a magnetic field. Can that magnetic field reinforce the cause that produced it?

Lenz's rule states the fact (established by Maxwell's equations) that the feedback thus described is always negative. It never amplifies the cause. It always diminishes the cause.

Instances of positive feedback tend to lead to runaway processes such as when a microphone and a loudspeaker are too close together or when a patch of ice on a mountain slope starts to slide. Positive feedback is epitomized in chain reactions.

Any effects caused by induction counteracts the the magnetic flux change that produced them in the first place. This is Lenz's rule in a nutshell.

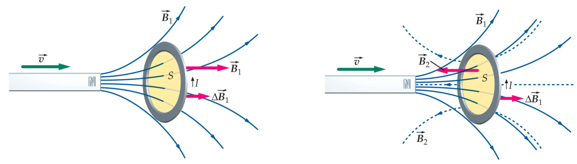
Now we are ready answer part (e) of the problem stated on page 10 of the previous lecture. A cw current causes a magnetic field circling the rod and the rails, up on the outside and down on the inside of the rectangle, thus counteracting the increasing flux through the rectangle. That is the negative feedback as predicted by Lenz's rule and dictated by Faraday's law.

On the next page we use Lenz's rule to predict the direction of an induced current in a different scenario.

Lenz's Rule (2)

In the situation shown below the current induced in the conducting ring generates a magnetic field whose flux counteracts the change in magnetic flux caused by the bar magnet.

- Moving the bar magnet closer to the ring increases the magnetic field $\vec{B}_1$ (solid field lines) through the ring by the amount $\Delta\vec{B}_1$.
- The resultant change in magnetic flux through the ring induces a current I in the direction shown.
- The induced current I, in turn, generates a magnetic field $\vec{B}_2$ (dashed field lines) in a direction that opposes the change of flux caused by the moving bar magnet.



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We have encountered this scenario before. A bar magnet is moved closer to a conducting ring. The magnetic flux through the ring changes because the magnetic field through the plane of the ring increases in strength.

Is the induced current counterclockwise (as indicated) or clockwise? Faraday's law gives the answer (see lecture 27). It predicts a ccw current as indicated on the slide.

Lenz's rule says that the induced current must provide a negative feedback. The magnetic field generated by the induced current must counteract the flux change caused by the moving bar magnet.

The ccw current direction indeed generates a magnetic field, as we have learned in lecture 23, such that the field lines inside the circle are directed left, thus opposing the field lines of the bar magnet. The feedback is negative because it diminishes the flux increase through the ring.

Suppose we move the bar magnet in the opposite direction. The flux will then decrease as the field near the ring gets weaker. The induced current will now be clockwise. The magnetic field it produces will be to the right inside the circle, counteracting the weakening field in the same direction of the bar magnet. The feedback is again negative.

So much about Lenz's rule. Let us return to Faraday's law. It can stand on its own. We complete this lecture with a series of applications.

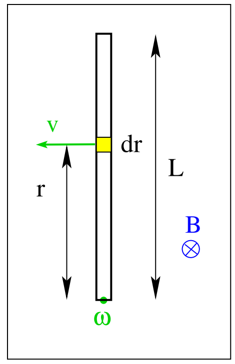
Magnetic Induction: Application (9)

Consider a conducting rod of length L rotating with angular velocity ω in a plane perpendicular to a uniform magnetic field $\vec{B}$.

- Angular velocity of slice: ω
- Linear velocity of slice: $v = \omega r$
- EMF induced in slice: $d\mathcal{E} = Bvdr$
- Slices are connected in series.
- EMF induced in rod:

$$\mathcal{E} = \int_0^L Bv\,dr = B\omega \int_0^L r\,dr$$

$$\Rightarrow\ \mathcal{E} = \frac{1}{2}B\omega L^2 = \frac{1}{2}Bv_0L, \quad v_0 = \omega L$$



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This scenario is best analyzed as an application of motional EMF. What is different from the scenario presented on the first page of lecture 27 is that not all parts of the rod move at the same speed.

let us think of the rotating rod as an array infinitesimal slices connected in series. The instantaneous velocity v of each slice depends on its radial distance r from the pivot.

The EMF across each slice is as if its motion were translational. Adding up the EMFs across successive slices amounts to performing an integral as shown.

If the rod rotates with angular velocity ω, its outer end travels with speed $v_0 = \omega L$. The induced EMF,

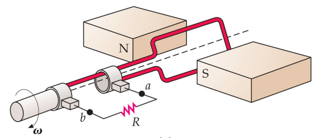
$$\mathcal{E} = \frac{1}{2}Bv_0L,$$

is half the induced EMF in a rod of length L travels translationally with the same speed v_0.

In the rotating rod, the average speed of segments is $v_0/2$, which explains the result in a different way.

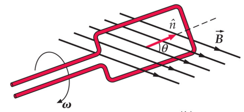
Which end of the rod is at the higher potential? The quickest answer comes from the direction of magnetic force, $\vec{F} = q\vec{v} \times \vec{B}$, acting on mobile charge carriers inside the rod. Positive charge carriers moving $\leftarrow$ in a magnetic field directed $\otimes$ are pushed $\downarrow$. Therefore, high potential is at the pivot.

AC Generator



(a)

- Area of conducting loop: A
- Number of loops: N
- Area vector: $\vec{A} = A\hat{n}$
- Magnetic field: $\vec{B}$
- Angle between vectors $\vec{A}$ and $\vec{B}$: $\theta = \omega t$
- Magnetic flux: $\Phi_B = N\vec{A} \cdot \vec{B} = NAB\cos(\omega t)$
- Induced EMF: $\mathcal{E} = -\frac{d\Phi_B}{dt} = \underbrace{NAB\omega}_{\mathcal{E}_{max}}\sin(\omega t)$



(b)

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Alternating-current (ac) circuits will come up as a topic of this course soon. They are driven by power sources that deliver an oscillating EMF. Our electrical outlets are such sources.

The principle of ac power generation is a very simple application of Faraday's law. A mechanically powered turbine forces a current loop to rotate in a static magnetic field.

The steady rotation of the loop with angular velocity ω changes the angle between its area vector $\vec{A}$ and the magnetic field $\vec{B}$ continually.

In consequence, the magnetic flux Φ_B through the loop changes periodically. It then follows from Faraday's law that the induced EMF $\mathcal{E}$ also changes periodically.

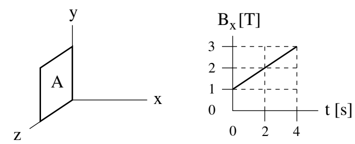
The underbraced quantity $\mathcal{E}_{max}$ is the maximum value of the ac voltage, called amplitude. Note that we get higher voltage if the loop is forced to rotate faster.

Intermediate Exam III: Problem #2 (Spring '06)

A conducting loop in the shape of a square with area $A = 4\text{m}^2$ and resistance $R = 5\Omega$ is placed in the yz-plane as shown. A time-dependent magnetic field $\mathbf{B} = B_x\hat{\mathbf{i}}$ is present. The dependence of B_x on time is shown graphically.

(a) Find the magnetic flux Φ_B through the loop at time $t = 0$.

(b) Find magnitude and direction (cw/ccw) of the induced current I at time $t = 2\text{s}$.



Choice of area vector: $\odot/\otimes$ $\Rightarrow$ positive direction = ccw/cw.

(a) $\Phi_B = \pm(1\text{T})(4\text{m}^2) = \pm 4\text{Tm}^2$.

(b) $\dfrac{d\Phi_B}{dt} = \pm(0.5\text{T/s})(4\text{m}^2) = \pm 2\text{V} \quad \Rightarrow \quad \mathcal{E} = -\dfrac{d\Phi_B}{dt} = \mp 2\text{V}$.

$\Rightarrow \; I = \dfrac{\mathcal{E}}{R} = \mp\dfrac{2\text{V}}{5\Omega} = \mp 0.4\text{A}$ (cw).

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When you read the problem statement on this slide, your thought process should be something like the following:

"In this application of Faraday's law, the relevant loop is the conducting square and the change of magnetic flux is caused by a change in magnetic field strength."

The magnetic field is uniform and pointing in the positive x-direction. The graph tells us that it increases linearly with time. We only need to know at what rate it increases, which is two Tesla in four seconds, i.e. 0.5T/s.

For the analysis, we now need to adopt a convention. The directions $\odot/\otimes$ stated on the slide mean $\pm\hat{\mathbf{i}}$.

Both possible conventions are carried through on the slide. The sign ($\pm$) of most quantities depend on the convention, but the physical result, namely the magnitude and direction of the induced current, does not. The positive current is cw, irrespective of convention.

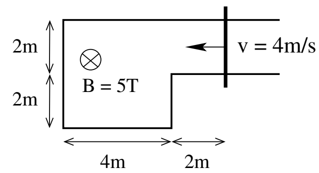
Intermediate Exam III: Problem #3 (Spring '07)

A conducting frame with a moving conducting rod is located in a uniform magnetic field as shown.

(a) Find the magnetic flux Φ_B through the frame at the instant shown.

(b) Find the induced emf $\mathcal{E}$ at the instant shown.

(c) Find the direction (cw/ccw) of the induced current.



Solution:

(a) $\Phi_B = \vec{A} \cdot \vec{B} = \pm(20\text{m}^2)(5\text{T}) = \pm 100\text{Wb}.$

(b) $\mathcal{E} = -\frac{d\Phi_B}{dt} = \pm(5\text{T})(2\text{m})(4\text{m/s}) = \pm 40\text{V}.$

(c) clockwise.

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Here your reading of the problem statement should trigger the following thought process:

"The relevant loop is delineated by the (rigid) frame and the (moving) rod. The (uniform) magnetic field $\vec{B}$ is constant in both magnitude and direction. The change in flux is caused by the shrinking area of the loop."

The two signs on the slide pertain to different conventions. The upper (lower) sign has $\vec{A}$ directed $\otimes$ ($\odot$), implying that the positive loop direction cw (ccw).

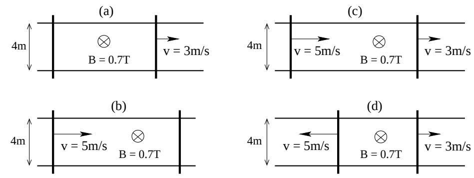
The loop area at the instant shown, which we need to know for part (a), consists of two squares with sides 4m and 2m, respectively.

For part (b) we need to know the rate at which the area shrinks in size. At the instant shown, this rate is equal to the length of rod between the two contact points with the frame times the velocity of the rod (the rate at which it changes position).

Note again that the physical result, here exemplified by the direction of the induced current, is independent of the convention.

Unit Exam III: Problem #3 (Spring '09)

A pair of rails are connected by two mobile rods. A uniform magnetic field B directed into the plane is present. In the situations (a), (b), (c), (d), one or both rods move at constant velocity as shown. The resistance of the conducting loop is $R = 0.2\Omega$ in each case. Find magnitude I and direction (cw/ccw) of the induced current in each case.



Solution:

(a) $|\mathcal{E}| = (3\text{m/s})(0.7\text{T})(4\text{m}) = 8.4\text{V}, \quad I = \dfrac{8.4\text{V}}{0.2\Omega} = 42\text{A}$ ccw

(b) $|\mathcal{E}| = (5\text{m/s})(0.7\text{T})(4\text{m}) = 14\text{V}, \quad I = \dfrac{14\text{V}}{0.2\Omega} = 70\text{A}$ cw

(c) $|\mathcal{E}| = (5\text{m/s} - 3\text{m/s})(0.7\text{T})(4\text{m}) = 5.6\text{V}, \quad I = \dfrac{5.6\text{V}}{0.2\Omega} = 28\text{A}$ cw

(d) $|\mathcal{E}| = (5\text{m/s} + 3\text{m/s})(0.7\text{T})(4\text{m}) = 22.4\text{V}, \quad I = \dfrac{22.4\text{V}}{0.2\Omega} = 112\text{A}$ ccw

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This application has four simple variations of parts (b) and (c) on the previous page.

The convention adopted here is that the area vector $\vec{A}$ is directed $\otimes$, as the magnetic field is, and implying that the positive loop direction is cw.

The solution shown on the slide gives the magnitude of the induced EMF, which is equal to the magnitude of the rate at which the magnetic flux changes.

The rate of change of flux is equal to the magnetic field times the rate of change of area.

The rate of change of area is equal to the contact distance 4m times the difference in velocities between the two rods.

The magnitude of current follows from Ohm's law.

The current direction is cw (positive) if the induced EMF is positive, i.e. if the rate at which the flux changes is negative, i.e. if the the area shrinks. Otherwise the current direction is ccw (negative).

Earlier in the course, we have stated that the direction of current I is a matter of choice because it is the flux of current density $\vec{J}$ through an open surface (e.g. a wire cross section) and thus depending on a choice of area vector. Depending on that choice, the current comes out positive or negative.

In the context of induced currents in applications of Faraday's law we have been predicting current directions, by which we always mean the *positive* current direction. This is not a matter of choice. The analysis either tells us if a current of chosen direction is positive or negative or if a positive current is directed one way or the other (e.g cw or ccw).

Magnetic Induction: Application (14)

Consider a conducting frame moving in the magnetic field of a straight current-carrying wire.

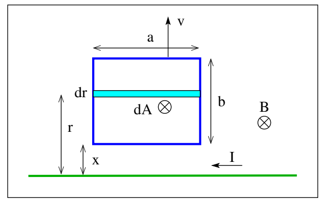
- magnetic field: $B = \frac{\mu_0 I}{2\pi r}$
- magnetic flux: $\Phi_B = \int \vec{B} \cdot \vec{A}, \quad dA = adr$

$$\Phi_B = \frac{\mu_0 I a}{2\pi} \int_x^{x+b} \frac{dr}{r} = \frac{\mu_0 I a}{2\pi} \Big[\ln(x+b) - \ln x \Big] = \frac{\mu_0 I a}{2\pi} \ln \frac{x+b}{x}$$

- induced EMF: $\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d\Phi_B}{dx}\frac{dx}{dt} = -\frac{d\Phi_B}{dx} v$

$$\mathcal{E} = -\frac{\mu_0 I a v}{2\pi} \left[\frac{1}{x+b} - \frac{1}{x} \right] = \frac{\mu_0 I a b v}{2\pi x (x+b)}$$

- induced current: $I_{ind} = \frac{\mathcal{E}}{R}$ clockwise



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This application of Faraday's law is a bit more advanced. The magnetic flux through the rectangular frame changes because the frame moves from a region of strong magnetic field (near the straight current that generates it) to a region of weaker field.

The first item states the magnitude of the magnetic field as a function of distance r from its source.

The instantaneous position of the frame is characterized by the coordinate x of the side closest to the wire. The calculation of the the magnetic flux Φ_B then amounts to a sum of thin strips across the frame, which is converted into an integral as worked out in the second item. Note that the flux is a function of instantaneous position $x(t)$.

When we calculate the induced EMF in the next item by using Faraday's law, we must employ the chain rule for the derivative with respect to time.

The result depends on the instantaneous position $x(t)$ and the instantaneous velocity $v(t) = dx/dt$.

Why is the induced current direction cw? If we use area vector $\vec{A}$ directed $\otimes$, then cw is the positive loop direction. In this convention, the magnetic flux is positive but decreases as the frame moves toward a weaker magnetic field. Hence we have $d\Phi_B/dt < 0$, implying that $\mathcal{E} > 0$. Now we remember that positive means cw.

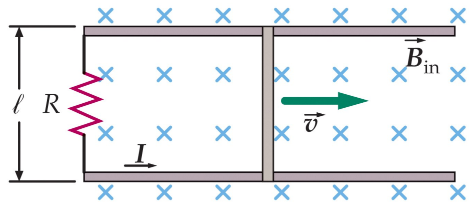
Magnetic Induction: Application (8)

Consider a rectangular loop of width ℓ in a uniform magnetic field $\vec{B}$ directed into the plane. A slide wire of mass m is given an initial velocity $\vec{v}_0$ to the right. There is no friction between the slide wire and the loop. The resistance R of the loop is constant.

(a) Find the magnetic force on the slide wire as a function of its velocity.

(b) Find the velocity of the slide wire as a function of time.

(c) Find the total distance traveled by the slide wire.



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(a) The induced (motional) EMF is $\mathcal{E} = vBl$. The induced current has magnitude $I = \mathcal{E}/R = vBl/R$ and is directed ccw.

The magnetic force on the current in the slide wire is $\vec{F} = I\vec{l} \times \vec{B}$ with $\vec{l}$ pointing $\uparrow$ for a ccw current. Hence the force is pointing $\leftarrow$ and has magnitude $F = B^2l^2v/R$. This has been the easy part.

(b) The magnetic force slows the moving slide wire down as dictated by Newton's second law, $F = ma$:

$$m\frac{dv}{dt} = -\frac{B^2l^2}{R}v \quad \Rightarrow \quad \int_{v_0}^{v}\frac{dv}{v} = -\frac{B^2l^2}{mR}\int_0^t dt$$

$$\Rightarrow \quad \ln\left(\frac{v}{v_0}\right) = -\frac{B^2l^2}{mR}t \quad \Rightarrow \quad v = v_0 \exp\left(-\frac{B^2l^2}{mR}t\right).$$

The velocity is gradually decreasing and approaching zero as $t \to \infty$.

(c) The position of the slide wire is calculated as the integral of its velocity:

$$x(t) = x_0 + \int_0^t v(t)dt = x_0 + v_0\int_0^t \exp\left(-\frac{B^2l^2}{mR}t\right)$$

$$= x_0 + \frac{v_0 mR}{B^2l^2}\left[-\exp\left(-\frac{B^2l^2}{mR}t\right)\right]_0^t$$

$$\Rightarrow \quad x(t) = x_0 + \frac{v_0 mR}{B^2l^2}\left[1 - \exp\left(-\frac{B^2l^2}{mR}t\right)\right].$$

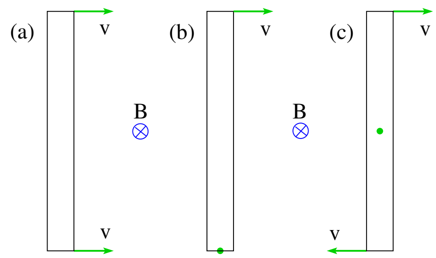
The total distance traveled by the slide wire is finite even though it reaches there only after an infinite time:

$$\Delta x = x(\infty) - x(0) = \frac{v_0 mR}{B^2l^2}.$$

Magnetic Induction: Application (1)

Consider three metal rods of length $L = 2\text{m}$ moving translationally or rotationally across a uniform magnetic field $B = 1\text{T}$ directed into the plane.
All velocity vectors have magnitude $v = 2\text{m/s}$.

- Find the induced EMF $\mathcal{E}$ between the ends of each rod.



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This is the quiz for lecture 28.

Here we pick up the theme of page 4. State three numbers in Volts.