PHY204 Lecture 30 [rln30]

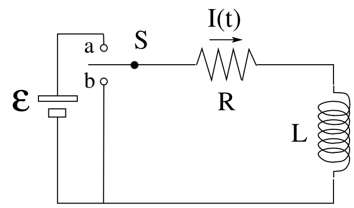
RL Circuit: Fundamentals

Specifications:

- $\mathcal{E}$ (emf)
- R (resistance)
- L (inductance)

Switch S:

- a: current buildup
- b: current shutdown



Time-dependent quantities:

- $I(t)$: instantaneous current through inductor
- $\frac{dI}{dt}$: rate of change of instantaneous current
- $V_R(t) = I(t)R$: instantaneous voltage across resistor
- $V_L(t) = L\frac{dI}{dt}$: instantaneous voltage across inductor

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This lecture is devoted to RL circuits, which contain resistors and inductors in addition to EMF sources.

Earlier we have discussed RC circuits, which contain resistors and capacitors. The mathematical analysis of RL circuits is strikingly similar to that of RC circuits, even though the physical attributes are quite different.

The slide on this page introduces the prototypical RL circuit. It has a switch with two settings.

Setting (a) connects the EMF source to a loop with a resistor and an inductor. In this setting a positive clockwise current is being built up.

Setting (b) disconnects the EMF source from the loop. The consequence is that the previously established current now gradually shuts down.

We have declared, by an arrow on the slide, that the current direction is clockwise. This is a smart choice the current during buildup and shutdown will come out to be positive.

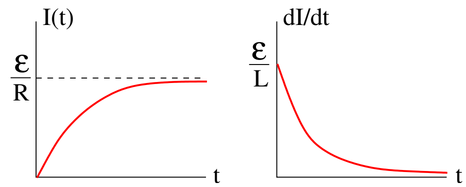
It is important to note that the current in the loop and the voltages across the resistor and inductor are all functions of time.

On the next two pages we analyze these time-dependent quantities for the two settings of the switch.

RL Circuit: Current Buildup in Inductor

- Loop rule: $\mathcal{E} - IR - L\frac{dI}{dt} = 0$
- Differential equation: $L\frac{dI}{dt} = \mathcal{E} - IR \quad \Rightarrow \quad \frac{dI}{dt} = \frac{\mathcal{E}/R - I}{L/R}$

  $$\int_0^I \frac{dI}{\mathcal{E}/R - I} = \int_0^t \frac{dt}{L/R} \quad \Rightarrow \quad -\ln\left(\frac{\mathcal{E}/R - I}{\mathcal{E}/R}\right) = \frac{t}{L/R} \quad \Rightarrow \quad \frac{\mathcal{E}/R - I}{\mathcal{E}/R} = e^{-Rt/L}$$
- Current through inductor: $I(t) = \frac{\mathcal{E}}{R}\left[1 - e^{-Rt/L}\right]$
- Rate of current change: $\frac{dI}{dt} = \frac{\mathcal{E}}{L}e^{-Rt/L}$



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Consider the current buildup process with the switch closed to setting (a). We have a one-loop circuit and invoke the loop rule for the analysis (see first item). We go around clockwise, first across the EMF source, then across the resistor, and finally across the inductor, to return to the starting point.

Each term represents a potential difference across one device. For the EMF it is what is on the label of the battery, for the resistor it is dictated by Ohm's law and for the inductor by Faraday's law.

We recognize that the loop rule is a differential equation for the function $I(t)$. It can be solved, as shown in the second item, by separation of variables. The lower integration boundaries represent the initial conditions: at time zero there is no current yet.

The explicit solution for $I(t)$ is stated in the third item and plotted at the bottom of the slide. It graphically describes the current buildup process. At the instant the switch is closed, the current starts growing gradually from zero. It reaches the value $\mathcal{E}/R$ asymptotically as $t \to \infty$.

Also shown, analytically and graphically, is the rate at which the current changes. That rate is largest at the beginning of the process and tapers off to zero as $t \to \infty$.

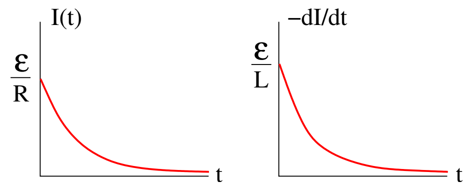
Apart from a change in scale the graph on the left represents the voltage, $V_R(t) = RI$, across the resistor and the graph on the right the voltage, $V_L(t) = L(dI/dt)$, across the inductor. The voltage across the resistor increases as the current increases. The voltage across the inductor decreases and approaches zero as the current becomes steady.

RL Circuit: Current Shutdown in Inductor

- Loop rule: $-IR - L\frac{dI}{dt} = 0$
- Differential equation: $L\frac{dI}{dt} + IR = 0 \quad \Rightarrow \quad \frac{dI}{dt} = -\frac{R}{L}I$

$$\Rightarrow \int_{\mathcal{E}/R}^{I} \frac{dI}{I} = -\frac{R}{L}\int_0^t dt \quad \Rightarrow \quad \ln\frac{I}{\mathcal{E}/R} = -\frac{R}{L}t \quad \Rightarrow \quad \frac{I}{\mathcal{E}/R} = e^{-Rt/L}$$

- Current: $I(t) = \frac{\mathcal{E}}{R}\, e^{-Rt/L}$
- Rate of current change: $\frac{dI}{dt} = -\frac{\mathcal{E}}{L}\, e^{-Rt/L}$



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Throwing the switch from setting (a) to setting (b) means disconnecting the EMF source from the loop. The loop rule is shown in the first item. We go around the loop clockwise first across the resistor and then across the inductor.

We again recognize the loop equation as a differential equation for $I(t)$. We solve that differential equation in the second item by the same method.

Note the different initial conditions used in the lower boundaries of the integrals. We have reset the clock to $t = 0$ when we reset the switch. At that instant, the current has its steady value $\mathcal{E}/R$ from the buildup process.

The explicit solution for $I(t)$ and its derivative are shown analytically and graphically on the slide.

In the shutdown process, we see the inertial property of inductance at work. It is the inductor which now drives the current through the resistor. It acts as an EMF source of sorts.

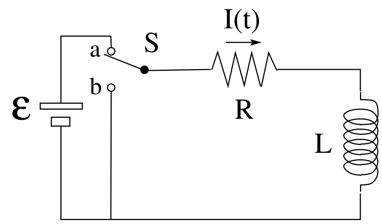
The voltage $L(dI/dt)$ across the inductor switches sign between current buildup and current shutdown. It now pushes the current forward. The current slows down as it is being pushed across the resistor.

RL Circuit: Energy Transfer During Current Buildup

Loop rule: $IR + L\frac{dI}{dt} = \mathcal{E} \quad (I > 0,\ \frac{dI}{dt} > 0)$

- $I\mathcal{E}$: rate at which EMF source delivers energy
- $IV_R = I^2R$: rate at which energy is dissipated in resistor
- $IV_L = LI\frac{dI}{dt}$: rate at which energy is stored in inductor

Balance of energy transfer: $I^2R + LI\frac{dI}{dt} = I\mathcal{E}$



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On this page and the next, we examine the energy transfer between devices during current buildup and current shutdown.

The slide here restates the loop rule during the current buildup. Each term represents the voltage across one of the three devices that are connected in the loop in switch setting (a).

When we multiply each term with the same factor I, the equation remains valid but now has a different interpretation. All terms have the SI unit Watt [W=J/s], representing power, i.e. transfer of energy per time unit. The meaning of all three terms is spelled out in the three items on the slide.

Initially, when the current is still small, most of the power that the EMF source delivers goes into the inductor, where it is being stored. As the current becomes steady, the delivery of power becomes steady as well. But all of it is now being dissipated in the resistor, at the rate I^2R.

The power storage capacity of the inductor, $LI(dI/dt)$, goes down to zero when the current I becomes steady. The energy stored on the inductor has reached its full capacity: $U = \frac{1}{2}LI^2$. We know that already from earlier.

The mechanical analog of an EMF source building up a current to a steady value is a locomotive accelerating a train to a certain speed. During the acceleration, the power delivered by the locomotive is, in part, converted into kinetic energy of the entire train and, in part, dissipated via friction and air resistance.

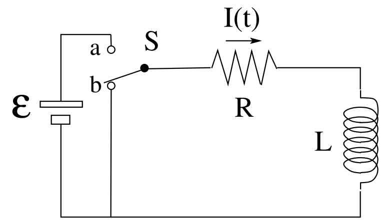
Once the train has reached traveling speed, the kinetic energy is no longer augmented, just as the energy in an inductor is no longer augmented when the current has become steady.

RL Circuit: Energy Transfer During Current Shutdown

Loop rule: $IR + L\frac{dI}{dt} = 0 \quad (I > 0,\ \frac{dI}{dt} < 0)$

- $IV_L = LI\frac{dI}{dt}$: rate at which inductor releases energy
- $IV_R = I^2R$: rate at which energy is dissipated in resistor

Balance of energy transfer: $I^2R + LI\frac{dI}{dt} = 0$



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In the setting (b) of the switch, the EMF source has been disconnected. We assume that at the beginning of the shutdown process, the current has the value $I(0) = \mathcal{E}/R$, the long-time asymptotic value of the buildup process.

There are now only two terms in the loop rule. When multiplied by the instantaneous current $I(t)$, each term again represents a power transfer. The meanings are spelled out in the two items on the slide.

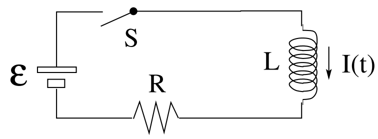
We have seen before that during the shutdown process it's the inductor that keeps the current flowing by virtue of its electromagnetic inertia. It can only accomplish this by releasing energy previously stored. That energy is needed to push the current through the resistor. The energy released from the inductor is being dissipated in the resistor, i.e. converted into a different form (e.g. heat).

The mechanical analog of the current shutdown process is a train slowing down due to friction between brake pads and wheels. The kinetic energy of the train is gradually converted into heat.

RL Circuit: Some Physical Properties

Specification of RL circuit by 3 device properties:

- $\mathcal{E}$ [V] (emf)
- R [Ω] (resistance)
- L [H] (inductance)



Physical properties of RL circuit during current buildup determined by 3 combinations of the device properties:

- $\frac{\mathcal{E}}{L} = \left.\frac{dI}{dt}\right|_{t=0}$: initial rate at which current increases
- $\frac{\mathcal{E}}{R} = I(t=\infty)$: final value of current
- $L/R = \tau$: time it takes to build up 63% of the current through the circuit
  $[1 - e^{-1} = 0.632\ldots]$

tsl276

How do we characterize an RL circuit?

One way to characterize it is by stating its specifications: EMF $\mathcal{E}$, resistance R, and inductance L.

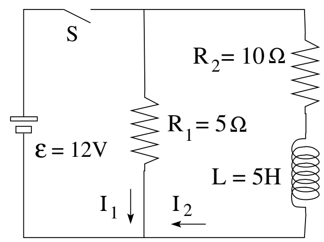
An alternative is to characterize it by key physical properties such as are captured in the three ratios, $\mathcal{E}/L$, $\mathcal{E}/R$, and L/R of the device properties with interpretations stated on the slide.

This characterization tells us how to adjust the device properties if we aim for particular values of any of the characteristic ratios.

RL Circuit: Application (8)

In the circuit shown the switch has been open for a long time.
Find the currents I_1 and I_2

- just after the switch has been closed,
- a long time later,
- as functions of time for $0 < t < \infty$.



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We complete this lecture with a series of circuits that contain resistors and inductors in a variety of combinations. Additional applications can be found among the exam 3 slides.

(a) When we connect the battery by closing the switch, resistor 1 experiences at once a voltage $\mathcal{E} = 12$V, which at once generates a current $I_1 = \mathcal{E}/R_1$. The series combination of inductor and resistor 2 also experiences 12V at once. However, we know that the current through an inductor can only change gradually. Hence the answers are

$$I_1 = \frac{12\text{V}}{5\Omega} = 2.4\text{A}, \quad I_2 = 0.$$

Nothing changes for resistor 1. In the outer loop we have a current buildup in process.

(b) Once current I_2 has grown to a steady value, the inductor becomes invisible in the sense that the voltage $L(dI/dt)$ across it is zero. Resistor 2 is now, effectively, in parallel with resistor 1. Hence we have

$$I_1 = \frac{12\text{V}}{5\Omega} = 2.4\text{A}, \quad I_2 = \frac{12\text{V}}{10\Omega} = 1.2\text{A}.$$

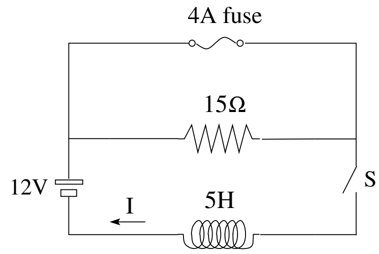
(c) The current I_2 undergoes the buildup process discussed before:

$$I_2(t) = \frac{\mathcal{E}}{R_2}\left(1 - e^{-R_2 t/L}\right).$$

Between $t = 0$ and $t = \infty$, I_2 gradually changes between the two values determined in parts (a) and (b).

In the circuit shown the switch S is closed at time $t = 0$.

(a) Find the current I as a function of time for $0 < t < t_F$, where t_F marks the instant the fuse breaks.
(b) Find the current I as a function of time for $t > t_F$.



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This circuit features a fuse in addition to an EMF source, a resistor, and an inductor. The fuse is a device that has zero resistance for as long as the current stays below a certain threshold value, here 4A. When the current reaches 4A, it breaks, i.e. it opens the branch. Intact fuses have zero resistance, broken fuses have infinite resistance.

What happens when we close the switch with the fuse intact?

While the fuse is intact, the resistor is short-circuited by the branch with the fuse. Both devices are in parallel and thus have the same voltage across. That voltage is zero across the fuse, hence also across the resistor. In consequence, there is no current through the resistor. We effectively have a one-loop circuit with EMF source, the inductor, and the intact fuse in series. The loop rule thus becomes,

$$\mathcal{E} - L\frac{dI}{dt} = 0,$$

from which we calculate the current as a function of time as follows:

$$\frac{dI}{dt} = \frac{\mathcal{E}}{L} = \frac{12\text{V}}{5\text{H}} = 2.4\text{A/s} = \text{const.} \quad \Rightarrow \; I(t) = (2.4\text{A/s})t.$$

The current increase linearly in time from zero. It reaches the values 4A at time $t_F = (4\text{A})/(2.4\text{A/s}) = 1.67\text{s}$. At this moment, the fuse breaks.

From that instant on, we are dealing with a different one-loop circuit. The open fuse now forces the 4A current already flowing through the EMF source and the inductor to also flow through the resistor. The new loop rule reads,

$$\mathcal{E} - RI - L\frac{dI}{dt} = 0.$$

This is a differential equation for the function $I(t)$, which we can solve by separating the variables I and t as follows:

$$\Rightarrow \ \frac{dI}{dt} = \frac{\mathcal{E}/R - I}{L/R} \quad \Rightarrow \ \frac{dI}{I - \mathcal{E}/R} = -\frac{R}{L}\, dt.$$

Next we integrate both sides. The integrations begin at time $t_F = 1.67$s, when the current has the value $I_F = 4$A.

$$\int_{I_F}^{I} \frac{dI'}{I' - \mathcal{E}/R} = -\int_{t_F}^{t} \frac{R}{L}\, dt' \quad \Rightarrow \ \ln\left(\frac{I - \mathcal{E}/R}{I_F - \mathcal{E}/R}\right) = -\frac{R}{L}(t - t_F).$$

As is custom, we have renamed the integration variables in order to distinguish them from the values at the upper boundaries. Solving the last expression for I yields the explicit expression,

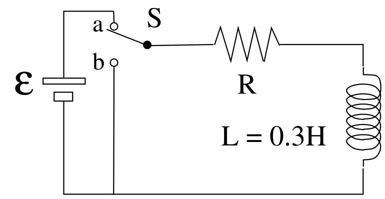
$$I(t) = \frac{\mathcal{E}}{R} + \left(I_F - \frac{\mathcal{E}}{R}\right) \exp\left(-\frac{R}{L}(t - t_F)\right).$$

Note that the final current with the fuse intact is the same as the initial current with the fuse broken, namely $I_F = 4$A. At $t > t_F$ the current gradually decreases and settles at the steady-state value $I(\infty) = \mathcal{E}/R = 0.8$A.

RL Circuit: Application (6)

In the RL circuit shown the switch has been at position a for a long time and is thrown to position b at time $t = 0$. At that instant the current has the value $I_0 = 0.7\text{A}$ and decreases at the rate $dI/dt = -360\text{A/s}$.

(a) Find the EMF $\mathcal{E}$ of the battery.
(b) Find the resistance R of the resistor.
(c) At what time t_1 has the current decreased to the value $I_1 = 0.2\text{A}$?
(d) Find the voltage across the inductor at time t_1.



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Here we have a simple RL circuit problem for which two device properties are unknown, namely $\mathcal{E}$ and R.

Not knowing any better, we begin with part (a). There is a steady current through the loop in the setting shown. The steady current produces zero voltage across the inductor. The loop rule gives us a relation with two unknowns:

$$\mathcal{E} - RI_0 = 0.$$

Let us, therefore, turn to part (b). At the instant the EMF source has been disconnected, the loop rule contains only one unknown:

$$-RI_0 - L\left(\frac{dI}{dt}\right)_0 = 0 \quad \Rightarrow \; R = -\frac{(0.3\text{H})(-360\text{A/s})}{0.7\text{A}} = 154\Omega.$$

Now we can solve part (a):

$$\mathcal{E} = (154\Omega)(0.7\text{A}) = 108\text{V}.$$

For part (c) we use the current shutdown expression and invert it:

$$I_1 = \frac{\mathcal{E}}{R}e^{-Rt_1/L} \quad \Rightarrow \; t_1 = -\frac{L}{R}\ln\left(\frac{I_1 R}{\mathcal{E}}\right) = 2.44\text{ms}.$$

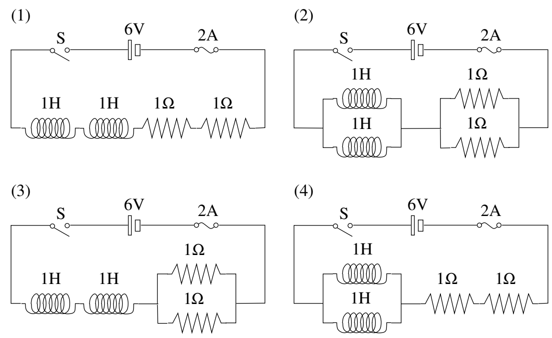
In part (d) we return to the loop rule used in part (b), but now for $t = t_1$ where $I = I_1$:

$$-RI_1 - L\left(\frac{dI}{dt}\right)_1 = 0 \quad \Rightarrow \; V_L(t_1) = -L\left(\frac{dI}{dt}\right)_1 = RI_1 = (154\Omega)(0.2\text{A}) = 30.8\text{V}.$$

RL Circuit: Application (5)

Each RL circuit contains a 2A fuse. The switches are closed at $t = 0$.

- In what sequence are the fuses blown?



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What we are dealing with here is a current buildup process in four different RL circuits. Each circuit has an equivalent resistor in series with an equivalent inductor. In each circuit, the buildup process is terminated when the current reaches 2A. At this instant the fuse breaks. It happens at different times in the four circuits.

First we must calculate R_{eq} and L_{eq} for each circuit, which is simple enough. Check page 10 of lecture 29.

Next we use the current expression from page 2 of this lecture, set $I(t_F) = I_{max} = 2\text{A}$ and solve it for t_F:

$$I_{max} = \frac{\mathcal{E}}{R_{eq}}\left[1 - \exp\left(-\frac{R_{eq}}{L_{eq}}\,t\right)\right] \quad \Rightarrow \quad t_F = -\frac{L_{eq}}{R_{eq}}\ln\left(1 - \frac{I_{max}R_{eq}}{\mathcal{E}}\right).$$

The four values of t_F come out as follows:

(1) $t_F = 1.099\text{s}$

(2) $t_F = 0.182\text{s}$

(3) $t_F = 0.729\text{s}$

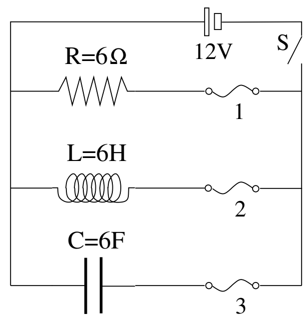
(4) $t_F = 0.275\text{s}$

The ranking follows directly.

RL Circuit: Application (1)

Each branch in the circuit shown contains a 3A fuse. The switch is closed at time $t = 0$.

(a) Which fuse is blown in the shortest time?
(b) Which fuse lasts the longest time?



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This is the quiz for lecture 30.