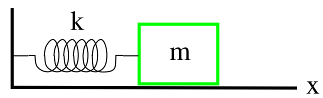
PHY204 Lecture 31 [rln31]

Mechanical Oscillator

- law of motion: $F = ma, \quad a = \frac{d^2x}{dt^2}$
- law of force: $F = -kx$
- equation of motion: $\frac{d^2x}{dt^2} = -\frac{k}{m}x$
- displacement: $x(t) = x_{max}\cos(\omega t)$
- velocity: $v(t) = -\omega x_{max}\sin(\omega t)$
- angular frequency: $\omega = \sqrt{\frac{k}{m}}$
- kinetic energy: $K = \frac{1}{2}mv^2$
- potential energy: $U = \frac{1}{2}kx^2$
- total energy: $E = K + U = \text{const.}$



tsl294

This lecture is devoted to circuits with capacitors, inductors, and resistors in different combinations. We are already familiar with RC circuits (charging and discharging of capacitors) and RL circuits (current buildup and shutdown). We shall see that LC circuits are electromagnetic oscillators.

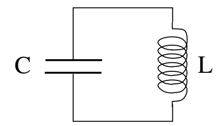
In many instances, it is illuminating to see and understand the analogies between certain mechanical and electromagnetic phenomena. We begin with a review of the mechanical oscillator, which has, as mentioned, its electromagnetic counterpart in the LC circuit.

The slide shows a block of mass m on a frictionless surface, attached to a wall by a spring with stiffness k. The block undergoes harmonic oscillations. The equation of motion (Newton's second law) [first item] with the particular elastic force in action [second item] is a familiar differential equation for the position variable [third item] with a familiar solution [fourth item]. The velocity $v(t)$ [fifth item] is the derivative of the position function $x(t)$.

There are two forms of energy in play: kinetic energy and potential energy. The former depends on velocity and the latter on position. Both quantities are function of time. The total (mechanical) energy is conserved, implying that during the oscillations, energy is converted back and forth between kinetic and potential.

When the spring is instantaneously relaxed ($x = 0$), the energy is all kinetic and when the block is instantaneously at rest ($v = 0$), it is all potential.

- loop rule: $\frac{Q}{C} + L\frac{dI}{dt} = 0,\ I = \frac{dQ}{dt}$
- equation of motion: $\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q$
- charge on capacitor: $Q(t) = Q_{max}\cos(\omega t)$
- current through inductor: $I(t) = -\omega Q_{max}\sin(\omega t)$
- angular frequency: $\omega = \frac{1}{\sqrt{LC}}$
- magnetic energy: $U_B = \frac{1}{2}LI^2$ (stored on inductor)
- electric energy: $U_E = \frac{Q^2}{2C}$ (stored on capacitor)
- total energy: $E = U_B + U_E = \text{const.}$



tsl295

The LC circuit is an electromagnetic oscillator. We begin its analysis with the loop rule as usual [first item]. We recall that the current $I(t)$ is the derivative of the charge $Q(t)$ on the capacitor. This relationship is analogous to the velocity and position variables on the previous page.

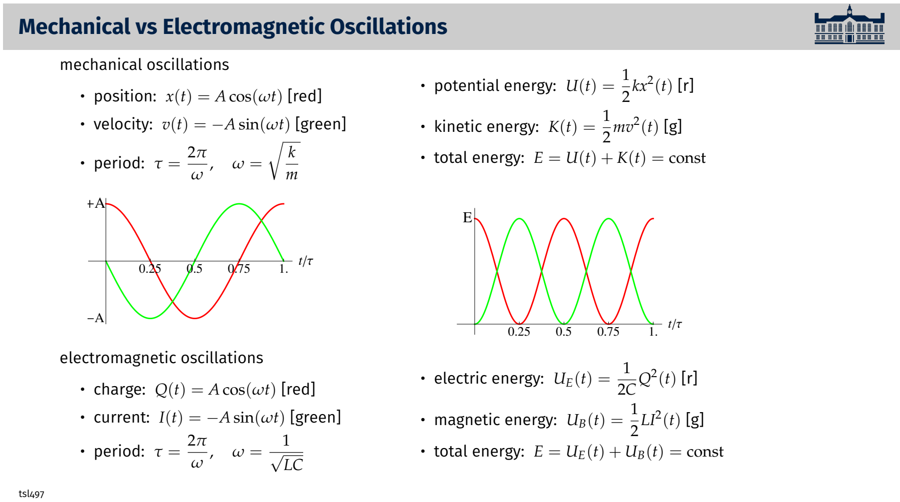
The loop rule then leads to a differential equation for the function $Q(t)$ that is mathematically equivalent to the one on the previous page. The solutions for $Q(t)$ and $I(t)$ follow in like manner.

Charge only accumulates on the capacitor but the current is everywhere except between the capacitor plates. However, for the energy accounting it's the current through the inductor that matters.

Keep the last clause in the first sentence of the previous paragraph in mind for later. There actually exist a sort of current between the capacitor plates. It is called *displacement current* and will play an important role in the full version of Ampère's law (stay tuned).

There are again two forms of energies that are being converted into each other back and forth with no loss. They are the magnetic energy located on the inductor and the electric energy located on the capacitor.

When the capacitor is fully charged, the current vanishes instantaneously as it changes direction. At these instants, the energy is all electric, stored in the electric field of the capacitor. The current reaches its maximum value when the capacitor is fully discharged. At those instants, the energy is all magnetic, stored in the magnetic field of the inductor.



This slide emphasizes the correspondence between attributes of the mechanical oscillator on top and the electromagnetic oscillator at the bottom.

The graph on the left shows two curves that represent the dynamical variable and its derivative. In the mechanical realization of the oscillator, they are the position of the block and its velocity. In the LC circuit, they are the charge on the capacitor and the current through the inductor.

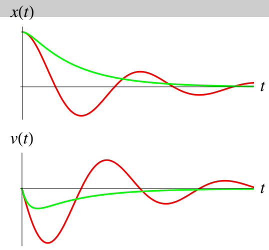
The two curves in the graph on the right show the two forms of energy. They are the potential energy and kinetic energy in the mechanical oscillator. In the electromagnetic oscillator, they are the electric energy and magnetic energy.

Note that during one full period of the oscillation, each form of energy reaches its maximum twice. The kinetic energy has a maximum when the block moves with maximum velocity in one or the other direction. The potential energy has a maximum when the spring is maximally compressed or maximally stretched.

Likewise, the electric energy has a maximum when the capacitor is fully charged one way and then again with positive and negative charges interchanged. The magnetic energy has its peaks when the current reaches a maximum flowing in one or the other direction.

Mechanical Oscillator with Damping

- law of motion: $F = ma, \quad a = \frac{d^2x}{dt^2}$
- law of force: $F = -kx - bv, \quad v = \frac{dx}{dt}$
- equation of motion: $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$



Solution for initial conditions $x(0) = A$, $v(0) = 0$:

(a) **underdamped motion**: $b^2 < 4km$

$$x(t) = Ae^{-bt/2m}\left[\cos(\omega' t) + \frac{b}{2m\omega'}\sin(\omega' t)\right] \quad \text{with} \quad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

(b) **overdamped motion**: $b^2 > 4km$

$$x(t) = Ae^{-bt/2m}\left[\cosh(\Omega' t) + \frac{b}{2m\Omega'}\sinh(\Omega' t)\right] \quad \text{with} \quad \Omega' = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

tsl299

The analogy between mechanical and electromagnetic oscillators still holds in the face of energy dissipation. In the mechanical case we add a damping force to the elastic force [second item]. This adds a term to the differential equation for the function $x(t)$ [third item].

Mathematically speaking this is a linear, homogeneous, second-order ordinary differential equation (ODE). Methods of analysis are typically discussed in higher-level courses. Here we just pull the solution shown on the slide out of a magician's hat. That hat could be a software such Mathematica or Matlab.

Note that the expression is different for the two cases of weak and strong coupling. There is yet a different solution at the border between the two regime (called critical damping). The graphs show that oscillations only persist in the underdamped case but the periodicity is no longer present.

If you are unfamiliar yet with hyperbolic function, it is enough for now to know that the hyperbolic cosine and sine functions are linear combinations of exponential functions:

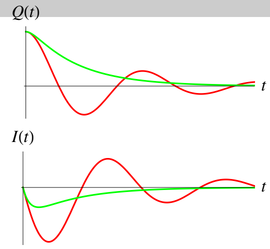
$$\cosh x \doteq \frac{1}{2}[e^x + e^{-x}], \quad \sinh x \doteq \frac{1}{2}[e^x - e^{-x}].$$

Part of their usefulness is that the two are derivatives of each other.

The sum of kinetic and potential energy is no longer a constant. It is now a decreasing function of time (not shown graphically). That does not mean that energy is not conserved. It simply means that mechanical energy is converted into yet a different form of energy (heat caused by friction).

Damped Electromagnetic Oscillator (RLC Circuit)

- loop rule: $RI + L\frac{dI}{dt} + \frac{Q}{C} = 0,\ I = \frac{dQ}{dt}$
- equation of motion: $\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0$



Solution for initial conditions $Q(0) = Q_{max},\ I(0) = 0$:

(a) **underdamped motion**: $R^2 < \frac{4L}{C}$

$$Q(t) = Q_{max}\, e^{-Rt/2L}\left[\cos(\omega' t) + \frac{R}{2L\omega'}\sin(\omega' t)\right] \quad \text{with} \quad \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

(b) **overdamped motion**: $R^2 > \frac{4L}{C}$

$$Q(t) = Q_{max}\, e^{-Rt/2L}\left[\cosh(\Omega' t) + \frac{R}{2L\Omega'}\sinh(\Omega' t)\right] \quad \text{with} \quad \Omega' = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

tsl300

Adding damping to the mechanical oscillator corresponds to adding a resistor to the LC circuit. The mathematical equivalence is quite evident. In the loop rule we now include a term RI.

The differential equation for the function $Q(t)$ of the RLC circuit has the same structure as that for the function $x(t)$ in the mechanical oscillator. Only the parameters are different. The correspondences are as follows:

$$x(t) \mathrel{\widehat{=}} Q(t), \quad v(t) \mathrel{\widehat{=}} I(t), \quad m \mathrel{\widehat{=}} L, \quad b \mathrel{\widehat{=}} R, \quad k \mathrel{\widehat{=}} \frac{1}{C}.$$

The same correspondences can be used to transcribe the solutions. The curves in the graph are the same. Only the labeling has been changed.

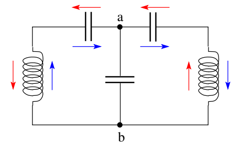
In the RLC circuit, the sum of magnetic and electric energy is no longer conserved. It decreases whenever a current flows. The presence of resistance converts some of the magnetic or electric energy into heat. In the underdamped regime, no energy is dissipated at those instants when the current changes direction.

Oscillator with Two Modes

Electromagnetic:

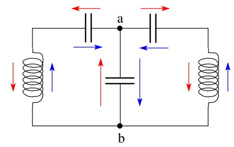
mode #1: $L\frac{dI}{dt} + \frac{Q}{C} + \frac{Q}{C} + L\frac{dI}{dt} = 0, \quad I = \frac{dQ}{dt}$

$\Rightarrow \frac{dI}{dt} = -\frac{Q}{LC} \quad \Rightarrow \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \frac{1}{\sqrt{LC}}$



mode #2: $L\frac{dI}{dt} + \frac{Q}{C} + \frac{2Q}{C} = 0, \quad I = \frac{dQ}{dt}$

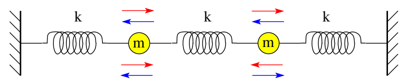
$\Rightarrow \frac{dI}{dt} = -\frac{3Q}{LC} \quad \Rightarrow \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \sqrt{\frac{3}{LC}}$



Mechanical:

mode #1: $\omega = \sqrt{\frac{k}{m}}$

mode #2: $\omega = \sqrt{\frac{3k}{m}}$



tsl498

Here we return to an electromagnetic oscillator without damping and its mechanical analogue. The LC circuit with two inductors and three capacitors can oscillate in two different ways (named modes), depending on what the initial state is.

In mode #1, the capacitor in the middle does not participate. It remains uncharged. The current oscillates as indicated by the arrows in the upper circuit diagram. The loop-rule analysis is worked out on the left as before.

In mode #2, there are two equal currents in the left and right branches. The current thus doubles in the middle. At one instant, the current directions are represented by blue arrows, half a period later by the red arrows. The loop-rule analysis is worked out for the loop on the left. It is identical for the loop on the right.

Note that the angular frequency of mode #2 is higher that of mode #1. When the LC circuit is launched from an arbitrary initial state, both modes are present simultaneously in superposition.

The same is true for the mechanical analogue shown a the bottom of the slide. When it oscillates in the slower mode, both masses move in the same direction all the time and the spring in the middle does not change its extension. The faster mode, by contrast, has the two masses move in opposite direction all the time, which now also compresses and stretches the spring in the middle.

RLC Circuit: Application (1)

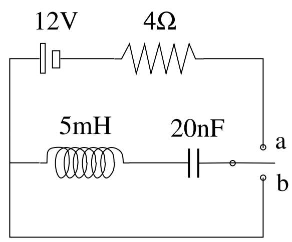
In the circuit shown the capacitor is without charge.
When the switch is closed to position a...

(a) find the initial rate dI/dt at which the current increases from zero,

(b) find the charge Q on the capacitor after a long time.
Then, when the switch is thrown from a to b...

(c) find the time t_1 it takes the capacitor to fully discharge,

(d) find the maximum current I_{max} in the process of discharging.



tsl438

When the switch is closed to a we have an RLC circuit including a battery. The loop rule includes four terms:

$$\mathcal{E} - L\frac{dI}{dt} - \frac{Q}{C} - RI = 0. \tag{1}$$

For part (a) we note that the capacitor is initially uncharged, $Q = 0$, and the current zero, $I = 0$, because no abrupt current change through the inductor is possible. That leaves only two terms in Eq. (1), from which infer the answer, $dI/dt = \mathcal{E}/L = 2400\text{A/s}$.

For part (b) we wait until the capacitor is fully charged and the current again zero, implying $I = 0$ and $dI/dt = 0$. From the two nonzero terms in Eq. (1) we infer the answer, $Q_{\text{max}} = C\mathcal{E} = 240\text{nC}$.

When we toggle from a to b, the battery and the resistor are disconnected. The new loop still contains the inductor and the capacitor. The current is instantaneously zero and the capacitor is fully charged. We have an LC oscillator running at angular frequency $\omega = 1/\sqrt{LC} = 1.00 \times 10^5\text{rad/s}$. The period is $\tau = 2\pi/\omega$ and the answer to part(c) is $t_1 = \frac{1}{4}\tau = 1.57 \times 10^5\text{s}$.

The quickest way to answer part (d) is to recall that $Q(t) = Q_{\text{max}}\cos(\omega t)$ with Q_{max} from part (b). Hence we have $I(t) = dQ/dt = -\omega Q_{\text{max}}\sin(\omega t)$, implying that $I_{\text{max}} = \omega Q_{\text{max}} = 24\text{mA}$.

Alternatively, we use energy conservation:

$$\frac{Q_{\text{max}}^2}{2C} = \frac{1}{2}LI_{\text{max}}^2 \quad \Rightarrow \quad I_{\text{max}} = \frac{Q_{\text{max}}}{\sqrt{LC}} = \omega Q_{\text{max}} = 24\text{mA}.$$

RLC Circuit: Application (2)

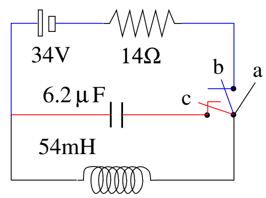
In the circuit shown the capacitor is without charge and the switch is in position a.

(i) When the switch is moved to position b we have an RL circuit with the current building up gradually: $I(t) = (\mathcal{E}/R)[1 - e^{-t/\tau}]$.

Find the time constant τ and the current I_{max} after a long time.

(ii) Then we reset the clock and move the switch from b to c with no interruption of the current through the inductor. We now have a an LC circuit: $I(t) = I_{max}\cos(\omega t)$.

Find the angular frequency of oscillation ω and the maximum charge Q_{max} that goes onto the capacitor periodically.



tsl499

When the switch is at a, there are no closed loops. Nothing moves. Closing the switch to b produces an RL circuit undergoing a current buildup process with the capacitor still disconnected. The slide quotes the a familiar result,

$$I(t) = \frac{\mathcal{E}}{R}\left[1 - e^{-t/\tau}\right].$$

(i) The time constant in an RL circuit with specs as shown on the slide is,

$$\tau = \frac{L}{R} = 3.86\text{ms}.$$

Once the current settles into a steady state, the inductor becomes invisible in the sense that the voltage across it goes down to zero. We then have, effectively, a loop with the EMF driving a current through the resistor. That current is the long-time limit in the buildup process:

$$I_{\max} = \lim_{t\to\infty} I(t) = \frac{\mathcal{E}}{R} = 2.43\text{A}.$$

(ii) Throwing the switch from b to c replaces the EMF source and the resistor by a capacitor while the current is flowing steadily. We now have an LC oscillator. The current with the clock reset at the instant the switch is thrown now oscillates as does the charge on the capacitor:

$$I(t) = I_{\max}\cos(\omega t) \quad \Rightarrow \; Q(t) = \int I(t)dt = \frac{I_{\max}}{\omega}\sin(\omega t),$$

with

$$\omega = \frac{1}{\sqrt{LC}} = 1.73 \times 10^3\text{rad/s}, \quad Q_{\max} = \frac{I_{\max}}{\omega} = 1.43\text{mC}.$$

RLC Circuit: Application (3)

In the circuit shown the capacitor is without charge and the switch is in position a.

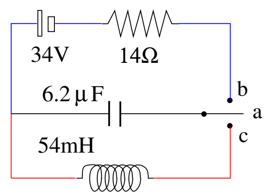
(i) When the switch is moved to position b we have an RC circuit with the capacitor being charged up gradually: $Q(t) = \mathcal{E}C[1 - e^{-t/\tau}]$.

Find the time constant τ and the charge Q_{max} after a long time.

(ii) Then we reset the clock and move the switch from b to c.

We now have a an LC circuit: $Q(t) = Q_{max}\cos(\omega t)$.

Find the angular frequency of oscillation ω and the maximum current I_{max} that flows through the inductor periodically.



tsl500

In this scenario, the inductor and the capacitor are effectively interchanged. We begin with the charging process of an RC circuit and then switch to an LC oscillator. The charge on the capacitor during the charging process is

$$Q(t) = \mathcal{E}C\left[1 - e^{-t/\tau}\right].$$

(i) Given the specs on the slide, the time constant and maximum charge become

$$\tau = RC = 86.8\mu\text{s}, \quad Q_{\max} = \lim_{t\to\infty} Q(t) = \mathcal{E}C = 2.11 \times 10^{-4}\text{C}.$$

(ii) When we throw the switch from b to c and reset the clock, the charge on the capacitor and the current through the inductor of the LC loop now become

$$Q(t) = Q_{\max}\cos(\omega t) \quad \Rightarrow \quad I(t) = \frac{dQ}{dt} = -\omega Q_{\max}\sin(\omega t),$$

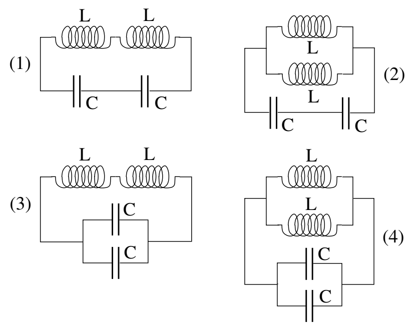
with

$$\omega = \frac{1}{\sqrt{LC}} = 1.73 \times 10^3\text{rad/s}, \quad I_{\max} = \omega Q_{\max} = 0.365\text{A}.$$

The LC circuit on this and the previous page are both launched with energy in them. In the circuit on this page, the initial energy is stored in the electric field of the charged-up capacitor. In the circuit on the previous page, it is stored in the magnetic field of the current-carrying inductor.

LC Circuit: Application (1)

Name the LC circuit with the highest and the lowest angular frequency of oscillation.



tsl296

This is the quiz for lecture 31.

Check the rules for equivalent capacitances and inductances as summarized in lecture 29.