PHY204 Lecture 35 [rln35]

Dynamics of Particles and Fields

Dynamics of Charged Particle:

- Newton's equation of motion: $\vec{F} = m\vec{a}$.
- Lorentz force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$.

Dynamics of Electric and Magnetic Fields:

- Gauss' law for electric field: $\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$.
- Gauss' law for magnetic field: $\oint \vec{B} \cdot d\vec{A} = 0$.
- Faraday's law: $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$, where $\Phi_B = \int \vec{B} \cdot d\vec{A}$.
- Ampère's law: $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I + \mu_0\epsilon_0 \frac{d\Phi_E}{dt}$, where $\Phi_E = \int \vec{E} \cdot d\vec{A}$.

Maxwell's equations: 4 relations between fields $(\vec{E}, \vec{B})$ and sources (q, I).

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The two goals we will be pursuing in this lecture are (i) to review and complete the laws that govern electricity and magnetism, and (ii) to set the table for a discussion of electromagnetic waves.

Most of the contents on the slide are familiar from earlier lectures. The dynamics of massive particles is governed by Newton's second law. Charged particles are subject to forces originating in electric and magnetic fields. Electric currents are streams of charged particles.

Electric charges q and electric currents I are called sources. An electric field $\vec{E}$ can be generated by a source q or by a time-varying magnetic field $\vec{B}$. A magnetic field $\vec{B}$ can be generated by a source I or by a time-varying electric field $\vec{E}$.

Sources and fields interact in a complex pattern of cause and effect:
– charged particles at rest or in motion generate fields,
– charged particles are accelerated by fields,
– one time-varying field generates the other field back and forth.

Note that Ampère's law has a second term on the right-hand side, which we have not yet discussed. Stay tuned for an explanation.

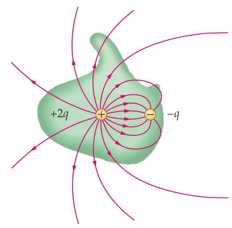
Gauss's Law for Electric Field

The net electric flux Φ_E through any closed surface is equal to the net charge Q_{in} inside divided by the permittivity constant ϵ_0:

$$\oint \vec{E} \cdot d\vec{A} = 4\pi k Q_{in} = \frac{Q_{in}}{\epsilon_0} \qquad \text{i.e.} \qquad \Phi_E = \frac{Q_{in}}{\epsilon_0} \qquad \text{with} \quad \epsilon_0 = 8.854 \times 10^{-12} \mathrm{C^2 N^{-1} m^{-2}}$$

The closed surface can be real or fictitious. It is called "Gaussian surface".
The symbol $\oint$ denotes an integral over a closed surface in this context.

- Gauss's law is a general relation between electric charge and electric field.
- In electrostatics: Gauss's law is equivalent to Coulomb's law.
- Gauss's law is one of four Maxwell's equations that govern cause and effect in electricity and magnetism.



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The first relation between source and field that we have encountered in this course is Gauss's law for the electric field. In its integral form, it relates the electric flux through a closed surface to the net charge inside.

There is a differential form of the same law, which relates source and field locally at the same point in space. The integral form is easier to visualize. The differential form requires some proficiency in vector calculus. The next page gives a glimpse of what is lying ahead in a higher-level course.

At the level of this introductory course, it suffices that we understand the following points:

- Electric field $\vec{E}$ is a vector, but electric flux Φ_E is not a vector.
- Electric flux is constructed from the surface integral of the dot product $\vec{E} \cdot d\vec{A}$ between two vectors.
- Electric flux can be calculated for open and closed surfaces.
- Gauss's law applies to closed surfaces (with an inside and an outside).
- The area vector $d\vec{A}$ points toward the outside of a closed surface.
- Gauss's law relates the electric flux through a closed surface to the net charge inside.
- For charges at rest, Gauss's law is equivalent to Coulomb's law.
- When charges are in motion, Gauss's law still holds exactly, Coulomb's law does not.

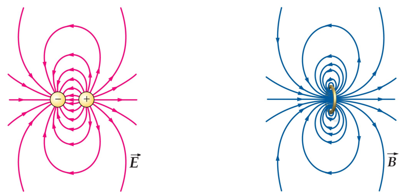
Gauss's Law for Magnetic Field

The net magnetic flux Φ_B through any closed surface is equal to zero:

$$\oint \vec{B} \cdot d\vec{A} = 0.$$

There are no magnetic charges. Magnetic field lines always close in themselves. No matter how the (closed) Gaussian surface is chosen, the net magnetic flux through it always vanishes.

The figures below illustrate Gauss's laws for the electric and magnetic fields in the context of an electric dipole (left) and a magnetic dipole (right).



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Gauss's law for the magnetic field is analogous. The important difference is that the source term (right-hand side) is zero. There are (presumably) no magnetic charges. This is an empirical fact.

The slide shows an electric dipole on the left and a magnetic dipole on the right. Only the electric dipole can be split into two electric monopoles (a positive and a negative electric charge). The magnetic dipole is realized as a ring of current.

There are other realizations of magnetic dipoles. Electrons, protons, and even neutrons have a magnetic dipole moment. None of the known realizations can be split into magnetic monopoles.

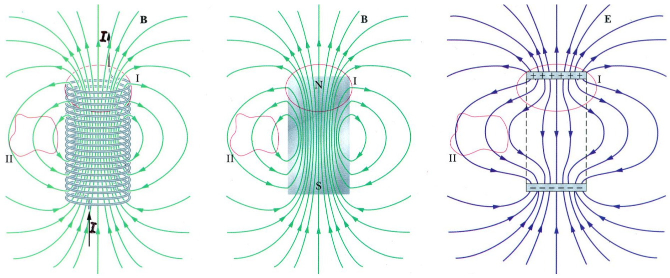
We can place closed surfaces into the field of the electric dipole (on the left) that produce positive flux or negative flux or zero flux. When applied to the magnetic field of the magnetic dipole (on the right), any closed surface will produce zero magnetic flux.

As in the electric case, there is a differential form of Gauss's law for the magnetic field. It states, that the components of the magnetic field must satisfy a certain local condition everywhere. In the jargon of vector analysis, it states that the divergence of the magnetic field vanishes:

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \text{i.e.} \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0.$$

The symbol $\vec{\nabla}$ is a differential operator named 'del' or 'nabla'. Here it operates on the magnetic field $\vec{B}$ in a particular way, indicated by '$\cdot$' familiar from the dot product.

$$\oint \vec{B} \cdot d\vec{A} = 0 \qquad \oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\epsilon_0}$$



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This slide shows two realizations of a magnetic dipole (left and center) and one realization of an electric dipole (right).

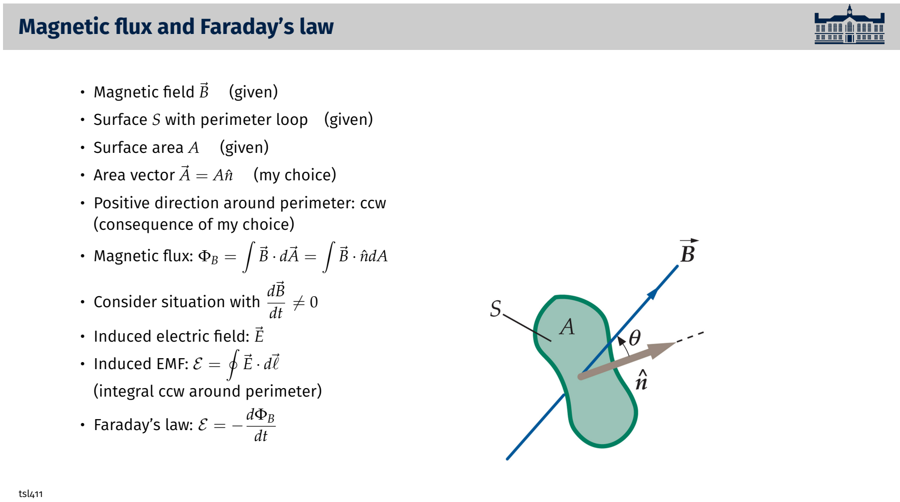
The magnetic dipole on the left is realized by a current-carrying solenoid.

The magnetic dipole in the center by a bar magnet of cylindrical shape. No current is flowing in the bar magnet. Here the magnetic field is generated by the magnetic moments of specific electrons in iron atoms that are aligned. When the bar is magnetized, these electron magnetic moments are aligned, pointing predominantly north (it's a long and complicated story).

The electric dipole is realized by two oppositely charged disks (seen in profile).

Note that the field lines outside the rectangular area (cylindrical shapes in profile) are very similar in all three cases. Inside the rectangle only the two realizations of the magnetic dipole have very similar field lines, whereas the field lines of the electric dipole are very different.

The main difference is, as mentioned earlier, that the magnetic field lines are closed in themselves, which the electric field lines are not. This observation is related to the fact that the magnetic flux through any closed surface is always zero. The electric flux through a closed surface can be positive, negative, or zero, depending on whether it envelops a positive charge, a negative charge, or no net charge.



Magnetic flux also plays a role in Faraday's law. What matters is the rate at which the magnetic flux through a loop of choice changes. This sentence is not nearly as innocent as it may sound.

A loop encloses a surface, real or imagined. If it is imagined, it can be imagined in multiple ways (as flat or variously curved shapes) with the same perimeter loop. The magnetic flux is independent of the choice. That follows from Gauss's law.

Whether the flux through the open surface with given perimeter loop is positive or negative depends on the choice of area vector. Linked to that choice is the choice of positive loop circulation (by right-hand rule).

Faraday's law relates the rate of magnetic flux through the loop thus established to the induced EMF around the loop. An induced EMF is not quite the same a the EMF (voltage, potential difference) provided by a battery.

What the changing magnetic flux primarily induces is an electric field with closed field lines. The induced electric field satisfies Gauss's law with zero sources (charges).

The induced EMF is equal to integral of that electric field around the loop in the positive loop direction previously identified. If the loop is a conducting wire, the induced EMF produces an induced current according to Ohm's law.

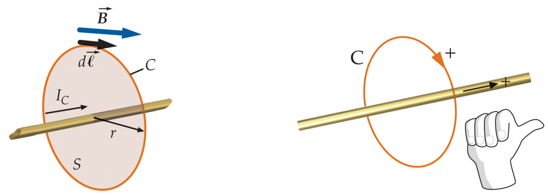
Ampère's Law (Restricted Version)

The circulation integral of the magnetic field $\vec{B}$ around any closed curve (loop) C is equal to the net electric current I_C flowing through the loop:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C, \qquad \text{with} \quad \mu_0 = 4\pi \times 10^{-7}\mathrm{Tm/A}$$

The symbol $\oint$ denotes an integral over a closed curve in this context.
Note: Only the component of $\vec{B}$ tangential to the loop contributes to the integral.

The positive current direction through the loop is determined by the right-hand rule.



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Loop integrals also feature in Ampère's law. The loop can again be real or imagined. The restricted version of that law, as introduced earlier and presented again on this slide, relates the loop integral of magnetic field to the net current flowing through the loop.

Choosing a positive direction of circulation around the loop determines (by right-hand rule) the positive current direction through the loop.

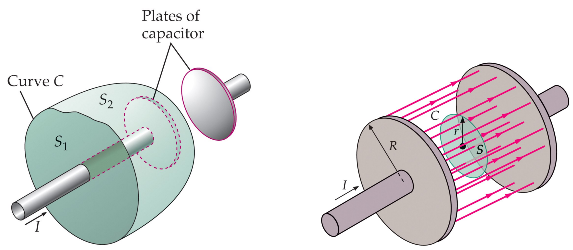
A completion of Ampère's law is necessary to satisfy charge conservation. The amount of charge in a given location can only change if some of it moves elsewhere. Moving charge is a current. Charge conservation thus requires that a relation (named continuity equation) between charge density and current density be satisfied.

Gauss's law for the electric field and plus charge conservation are only consistent with Ampère's law if the latter has the additional term shown on the slide of page 1.

The completion of Ampère's law might also be suggested by the following thought: If a changing magnetic flux through a loop induces an electric field, wouldn't a changing electric flux through a loop perhaps induce a magnetic field? Well, yes it does! Let us take a closer look.

Ampère's law (Full Version)

- Conduction current: I.
- Displacement current: $I_D = \epsilon_0 \frac{d\Phi_E}{dt}$.
- Ampère's law: $\oint \vec{B} \cdot d\vec{\ell} = \mu_0(I + I_D) = \mu_0 I + \mu_0\epsilon_0 \frac{d\Phi_E}{dt}$.



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Consider the setup shown on the left. A capacitor is being charged up by a current I in both connecting wires. Both currents flow in the same direction. The front (rear) plate is is being charged up positively (negatively). Positive charge flows onto the front plate and off the rear plate.

What do we mean when we say a current I flows through loop C. Do we mean crossing surface S_1 or, perhaps, surface S_2. The results would be inconsistent. It wouldn't help to insist on flat surfaces because not all loops are perimeters of flat surfaces. Also, if we shift the flat loop C toward the rear, the current through the flat surface S_1 suddenly stops and then picks up its original value.

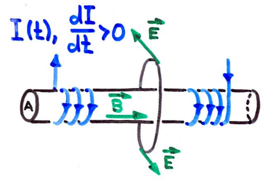
The problem is resolved by the realization that there are two types of currents in action: the familiar conduction current I and the yet unfamiliar and more abstract displacement current I_D.

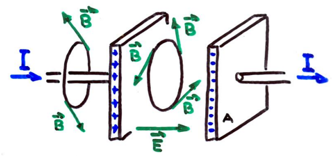
The nature of displacement current is illustrated in the image on the right. As the capacitor is being charge up, the electric field between the plates increases and so does the electric flux through the loop S. The product $\epsilon_0 d\Phi_E/dt$ has the SI unit of current. Hence the name displacement current.

The empirical evidence and internal consistency require that Ampère's law must be completed as shown in the third item. With this addition, it does no longer matter if we use surface S_1 or surface S_2 to determine the current through loop C in the image on the left. We have conduction current flowing through S_1 and displacement current flowing through S_2. The left-hand side of Ampère's law is unaffected by the choice of surface bounded by the same perimeter loop C.

Faraday's law and Ampère's law

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \qquad \oint \vec{B} \cdot d\vec{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$





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This slide illustrates an aspect of symmetry between Faraday's law and Ampère's law. At the same time, it illustrates an aspect of asymmetry between the two laws of nature.

On the left we have a solenoid and on the right a capacitor. The current in the winding of the solenoid is increasing, which produces a growing magnetic field $\vec{B}$ inside. The charge on the capacitor is increasing, which produces a growing electric field $\vec{E}$ between the plates.

Consider the three loops, one on the left and two on the right.

- The change of magnetic flux Φ_B through the loop around the solenoid generates an electric field in the region of the loop such that Faraday's law is satisfied.
- The conduction current flowing through the loop around the wire generates a magnetic field in the region of the loop such that Ampère's law is satisfied.
- The displacement current (rate of change of electric flux) flowing through the loop between the capacitor plates generates a magnetic field in the region of the loop such that Ampère's law is satisfied.

Note the opposite sign in the flux terms left and right, which produces induced fields in opposite direction.

Traveling Waves

Mechanical waves travel in some medium.
Examples: sound wave, violin string, surface water wave.
While the wave propagates, the medium undergoes periodic motion.

Distinguish:

(1) direction of wave propagation,
(2) direction in which medium moves.

Transverse wave: (1) and (2) are **perpendicular** to each other.

Longitudinal wave: (1) and (2) are **parallel** to each other.

Electromagnetic waves are transversely oscillating electric and magnetic fields.
Electromagnetic waves travel in the vacuum. There is no medium.

Waves transport energy and, in some cases, information, but not the medium itself (if there is a medium).

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In preparation of our next topic – electromagnetic waves – we have to become familiar with the wave phenomenon in general.

Mechanical waves are best understood as periodic perturbations of a continuous medium (a gas, liquid, or solid). The medium oscillates locally while the wave propagate.

In a longitudinal wave, such as a sound wave traveling through air, the direction of the oscillation of the medium is the same as the direction of wave propagation. The molecules oscillate back and forth locally, while regions of high density and low density travel at the speed of sound.

In a transverse wave, such as results when a violin string is being plucked, the two directions are perpendicular to each other. The atoms of the string oscillate perpendicular to axis of the string, while a pattern of transverse displacement travels back and forth along the string.

Water surface waves are a combination of longitudinal and transverse waves. Water molecules ascend and move forward with the wave crest, then descend and move backward in the wave trough, describing roughly a circular path.

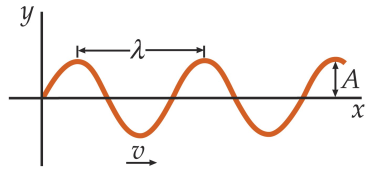
Electromagnetic waves are strictly transverse in nature and consist of electric and magnetic fields. The search for its medium yielded no results that would have made sense mechanically.

Einstein's theory of relativity proposed a new make-up of space and time that rendered the notion of a medium for electromagnetic waves superfluous.

Sinusoidal Transverse Traveling Wave

Wave function: $y(x,t) = A\sin(kx - \omega t)$

- $k = \frac{2\pi}{\lambda}$ (wave number)
- λ (wavelength)
- $\omega = \frac{2\pi}{T} = 2\pi f$ (angular frequency)
- $f = \frac{\omega}{2\pi} = \frac{1}{T}$ (frequency)
- T (period)
- $c = \frac{\lambda}{T} = \lambda f = \frac{\omega}{k}$ (wave speed)



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Waves of any kind are quantitatively described by wave functions. The wave function of a sinusoidal transverse wave is given in the first line on the slide. It is a function of two variables, position x and time t. The wave function has three parameters, amplitude A, wave number k, angular frequency ω.

The wave number k is related to the wavelength λ, the angular frequency ω to the frequency f and the period T as shown. The wave propagates one wavelength per period, which determines the wave speed c.

When viewed as a function of x at constant t, say $t = 0$, we have a snapshot of the wave form (shown on the slide),

$$y(x,0) = A\sin(kx).$$

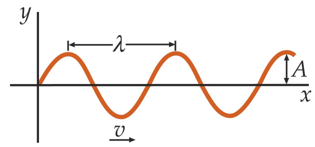
When viewed as a function of t at constant x, say $x = 0$, we follow the temporal oscillation of the medium (in a mechanical wave) or the temporal oscillations of the fields (in an electromagnetic wave),

$$y(0,t) = A\sin(-\omega t) = -A\sin(\omega t).$$

The wave is simultaneously periodic in space and in time. The spatial period is the wavelength λ and the temporal period is the period T.

Wave Equation

- $y(x,t) = A\sin(kx - \omega t)$ (displacement)
- $v(x,t) = \dfrac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t)$ (velocity)
- $a(x,t) = \dfrac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx - \omega t)$ (acceleration)
- $\dfrac{\partial y}{\partial x} = kA\cos(kx - \omega t)$ (slope of wave form)
- $\dfrac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx - \omega t)$ (curvature of wave form)
- $\dfrac{\partial^2 y/\partial t^2}{\partial^2 y/\partial x^2} = \dfrac{\omega^2}{k^2} = c^2$ (ratio of second derivatives)
- Wave equation: $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$



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Music is good if it sounds good. A physical phenomenon is a wave if it behaves like a wave (as outlined on the previous two pages).

A function $y(x,t)$ is a wave function if it satisfies the wave equation. The wave equation, shown in the last item of the slide, is a differential equation for the wave function. It contains a single parameter c, which is the wave speed.

This slide shows that the wave function given in the first item does indeed satisfy the wave equation. For this demonstration we have to differentiate twice with respect to the spatial variable x and twice with respect to the temporal variable t.

Now we know how to recognize a wave. The discovery of electromagnetic waves in the sense that certain phenomena were identified as such was an achievement of the late nineteenth century. Light, of course, has been seen for as long as eyes existed.

Our first goal in the next (and final) lecture is to show that the dynamics of electric and magnetic fields in empty space is governed by the wave equation. Hence the fields behave wave-like and travel with a particular speed, $c = 1/\sqrt{\epsilon_0\mu_0}$, which is, as it turns out, as fast as any physical thing can move.

Out of Africa

Four friends wish to cross a river at night on a narrow and treacherous bridge.

They have one flashlight.

To walk across the bridge it takes

- Gougouma 1minute,
- Ndakta 2 minutes,
- Maïtaïna 5 minutes,
- Kaïssebo 10 minutes.

The bridge supports not more than two persons simultaneously.

The use of the flashlight is essential on every trip across the bridge.

The duration of any trip is dictated by the slower person.

Describe the sequence of trips that minimizes the total time for the four friends to make it to the other side of the river.

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We conclude this lecture 35 with a puzzle from Chad, involving a river with menacing hippos, a fragile bridge, and four friends with a flashlight, who wish to cross the bridge at night. The four friends have Chadian names.

Two friends move forward, then one friend brings back the flashlight, then again two move forward and one back and, finally the last two forward, for a total of five crossings.

State the names for each crossing in sequence. If the total time is longer than 17 minutes you have to try harder.