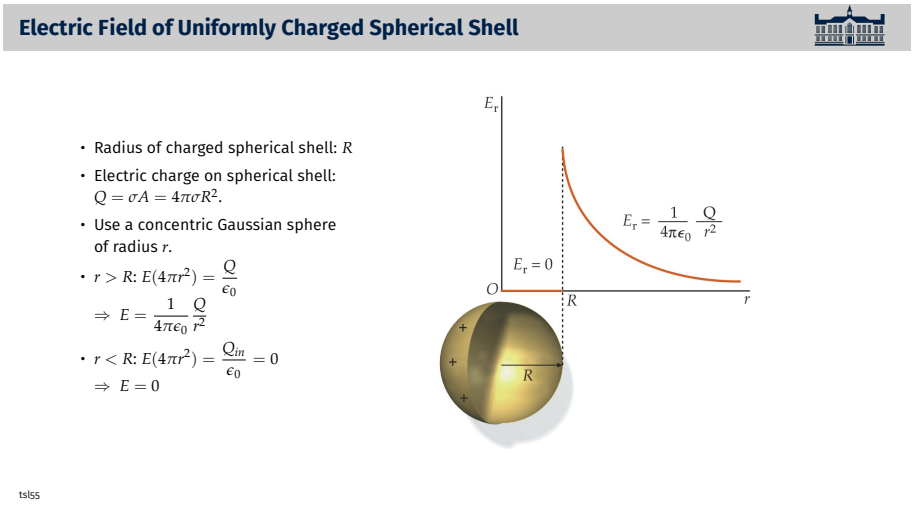
PHY204 Lecture 6 [rln6]



We begin this lecture with two applications of Gauss's law for the purpose of calculating the electric field of spherically symmetric charge distributions.

Here we consider a thin shell of radius R. It is uniformly charged, by which we mean that the charge per unit area σ is the same everywhere on the shell. We recall that the area is $A = 4\pi R^2$. Hence the total charge is $Q = \sigma A$.

Symmetry dictates that the electric field is radial and that its magnitude can only depend on the radius. Hence the electric flux through a (fictitious) Gaussian sphere of any radius r is the product of the field strength at that radius, $E(r)$, and the area of the Gaussian surface $4\pi r^2$.

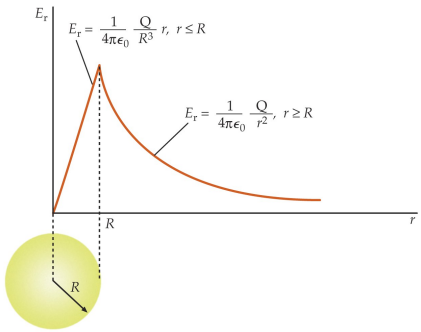
If $r > R$, then the spherical shell is inside the Gaussian surface. Therefore, the total charge inside is Q, the charge on the shell. Gauss's law, which relates the electric flux through the Gaussian surface to the net charge inside, thus produces an equation that can be solved for the field $E(r)$, which is the familiar Coulomb field.

If $r < R$, then the spherical shell is outside the Gaussian surface, which implies that there is no charge inside the Gaussian surface. Gauss's law then says that the flux through the Gaussian sphere, $4\pi r^2 E(r)$, is zero. The inevitable consequence is that $E(r)$ vanishes.

These results can be verified by using Coulomb's law. That calculation requires that we pick a field point either outside or inside the shell and then divide the charge distribution on the shell into rings, which have a given distance from the field point. Adding up the field contributions from all rings reproduces, with some effort, the above results.

Electric Field of Uniformly Charged Solid Sphere

- Radius of charged solid sphere: R
- Electric charge on sphere: $Q = \rho V = \frac{4\pi}{3}\rho R^3$.
- Use a concentric Gaussian sphere of radius r.
- $r > R$: $E(4\pi r^2) = \frac{Q}{\epsilon_0}$
  $\Rightarrow\ E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$
- $r < R$: $E(4\pi r^2) = \frac{1}{\epsilon_0}\left(\frac{4\pi}{3}r^3\rho\right)$
  $\Rightarrow\ E(r) = \frac{\rho}{3\epsilon_0}\,r = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\,r$



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Here we consider a solid sphere, again of radius R, but now with uniform volume charge density ρ. The volume of the sphere is $V = (4\pi/3)R^3$. The total charge is $Q = \rho V$.

The method of calculating the electric field $E(r)$ remains the same as described on the previous page. For distances $r > R$ from the center of the sphere, the result is the same. As long as the charge distribution is spherically symmetric and located inside the Gaussian sphere, we always recover the familiar Coulomb field of a point charge.

For values $r < R$, the electric field does not vanish as it does for the spherical shell. The reason for this difference is quite obvious. There is still electric charge positioned inside the Gaussian sphere, a fraction of the charge Q on the solid sphere. The amount of charge inside is equal to the charge density multiplied by the volume of the Gaussian sphere. The resultant field $E(r)$ is worked out in the last item on the slide.

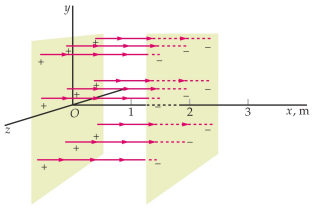
We see that the electric field vanishes only at the center of a uniformly charged solid sphere. The field strength increases linearly with radius between the center and the surface.

These results can again be verified by using Coulomb's law. In that calculation, we would slice the sphere into thin disks. We already know (from lecture 4) the field generated by a uniformly charged disk. Summing up the field contributions from disks that make up a solid sphere is mathematically more complex than this application of Gauss's law.

However, let us keep in mind that our elegant methodology only works if the symmetry conditions stated earlier apply.

Electric Field of Oppositely Charged Infinite Sheets

- Consider two infinite sheets of charge with charge per unit area $\pm\sigma$, respectively.
- The sheets are positioned at $x = 0$ and $x = 2$m, respectively.
- Magnitude of field produced by each sheet: $E = \dfrac{\sigma}{2\epsilon_0}$.
- Electric field at $x < 0$: $E_x = E_x^{(+)} + E_x^{(-)} = -\dfrac{\sigma}{2\epsilon_0} + \dfrac{\sigma}{2\epsilon_0} = 0.$
- Electric field at $0 < x < 2$m: $E_x = E_x^{(+)} + E_x^{(-)} = +\dfrac{\sigma}{2\epsilon_0} + \dfrac{\sigma}{2\epsilon_0} = \dfrac{\sigma}{\epsilon_0}.$
- Electric field at $x > 2$m: $E_x = E_x^{(+)} + E_x^{(-)} = +\dfrac{\sigma}{2\epsilon_0} - \dfrac{\sigma}{2\epsilon_0} = 0.$



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Now we switch focus from charge configurations with spherical symmetry to large, plane sheets with uniform charge per unit area σ spread over them. We already know that they are the source of uniform electric fields of strength $|\sigma|/2\epsilon_0$ on either side of them.

The field has the same magnitude and direction at all points on one or the other side of the sheet. If the sheet is positively (negatively) charged, the electric field is pointing away from it (toward it).

What is the electric field generated by two parallel sheets that are oppositely charged with uniform surface charge densities $+\sigma$ and $-\sigma$, respectively? The key to the answer is the superposition principle.

The sheets are positioned in a coordinate system as shown on the slide. The two sheets divide the space into three regions. On the left of both sheets or to the right of both sheets, each sheet produces a field of the same strength but opposite direction. Hence the resultant field vanishes.

In the third region, between the sheets, the two fields also have the same strength but now the directions are the same.

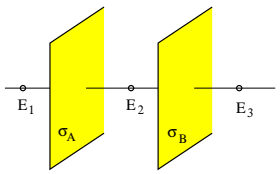
The result to remember is that oppositely charged parallel sheets produce a uniform electric field of strength $E = \sigma/\epsilon_0$ between them and zero field in the exterior regions. The electric field is pointing from the positively charged sheet toward the negatively charged sheet.

Unit Exam I: Problem #2 (Spring '09)

Consider two very large uniformly charged parallel sheets as shown.

The charge densities are $\sigma_A = +7 \times 10^{-12}\mathrm{Cm}^{-2}$ and $\sigma_B = -4 \times 10^{-12}\mathrm{Cm}^{-2}$, respectively.

Find magnitude and direction (left/right) of the electric fields $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$.



Solution:

$E_A = \dfrac{|\sigma_A|}{2\epsilon_0} = 0.40\mathrm{N/C}$ (directed away from sheet A).

$E_B = \dfrac{|\sigma_B|}{2\epsilon_0} = 0.23\mathrm{N/C}$ (directed toward sheet B).

$E_1 = E_A - E_B = 0.17\mathrm{N/C}$ (directed left).

$E_2 = E_A + E_B = 0.63\mathrm{N/C}$ (directed right).

$E_2 = E_A - E_B = 0.17\mathrm{N/C}$ (directed right).

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The two parallel sheets shown on the slide are assumed to be much larger than the image might suggest.

They are again uniformly charged, one positively and the other negatively. However, the charge densities, σ_A and σ_B, now have different magnitude.

We can still reason that the electric field is uniform in three regions, e.g. at the three field points indicated.

We first calculate the magnitude of the (uniform) fields E_A and E_B generated by the two sheets individually. The directions are away from the positively charged sheet and toward the negatively charged sheet.

In each region, we superpose the two fields thus generated, adopting the convention that a field directed right is counted positively and a field directed left is counted negatively.

Note that the field generated by sheet A is directed to the right on both sides of sheet B but only on one side of sheet A. Likewise, the field generated by sheet B is directed to the right on both sides of sheet A but only on one side of sheet B.

Unit Exam I: Problem #2 (Spring '12)

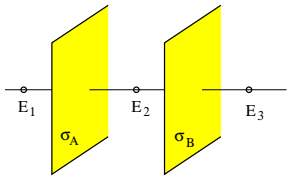
Two very large, thin, uniformly charged, parallel sheets are positioned as shown.

Find the values of the charge densities, σ_A and σ_B, if you know the electric fields $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$.

Consider two situations.

(a) $E_1 = 2\text{N/C}$ (directed left), $E_2 = 0$, $E_3 = 2\text{N/C}$ (directed right).

(b) $E_1 = 0$, $E_2 = 2\text{N/C}$ (directed right), $E_3 = 0$.



Solution:

(a) The two sheets are equally charged:
$\sigma_A = \sigma_B = 2\epsilon_0(1\text{N/C}) = 1.77 \times 10^{-11}\text{C/m}^2$.

(b) The two sheets are oppositely charged:
$\sigma_A = -\sigma_B = 2\epsilon_0(1\text{N/C}) = 1.77 \times 10^{-11}\text{C/m}^2$.

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Here we have essentially the same problem in reverse. Given are direction and magnitude of the electric field at the three field points indicated. We are being asked to find the (uniform) surface charge densities on the two sheets.

This is a straightforward application of Gauss's law. We use a can as Gaussian surface with flat areas A positioned across one or the other sheet.

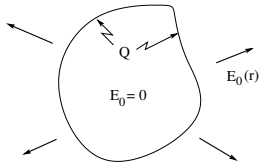
It is straightforward to determine the electric flux through the can. The only contributions arise from the flat surfaces. They are $\pm E_1 A$ and $\pm E_2 A$ for a can that cuts across sheet A, and $\pm E_2 A$ and $\pm E_3 A$ for a can that cuts across sheet B. The charge inside the can is either $\sigma_A A$ or $\sigma_B A$. The answers for σ_A and σ_B are given on the slide.

Note that fields pointing toward the outside of the can produce positive flux and fields pointing toward the inside produce negative flux.

This exercise comes with two scenarios, which are equally simple.

Charged Conductor at Equilibrium (1)

- Consider a conductor with excess charge Q in isolation.
- The mobile charges (electrons) are rearranged spontaneously until we have $\vec{E}_0 = 0$ everywhere inside the conductor.
- If $\vec{E}_0 = 0$ inside the conductor, then Gauss's law implies that there can be no net flux through any Gaussian surface that is inside the conductor.
- Hence there can be no net charge in any region inside the conductor.
- Hence all excess charge must be at the surface, where it produces an electric field $\vec{E}_0(\vec{r})$ on the outside only.



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In metallic conductors, some electrons are shared between atoms. These conduction electrons can move almost freely through the conducting material. Their motion is subject to collisions. Moving electrons represent an electric current. The collisions bring a current to a halt and the distribution of conduction electrons to equilibrium, unless the electrons experience a persistent force.

In lecture 1 we have briefly discussed how a conductor can be charged up positively or negatively by removing or adding conduction electrons.

How is any excess charge Q distributed across a conductor when it is at equilibrium? Mobile charges keep moving for as long as they experience an electric field. Electrostatic equilibrium in the presence of mobile charges thus implies zero electric field, $\vec{E}_0 = 0$, inside the conducting material.

If there were pockets of excess charge inside the bulk of the conductor, we could envelop any such pocket with a Gaussian surface. That surface would be embedded in the conducting material where there is no field. Hence there is no flux through that surface, which, according to Gauss's law, contradicts the claim that the surface envelops a pocket of excess charge.

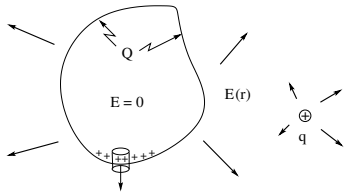
The only other place where the excess charge can be located is on the surface of the conductor.

The equilibrium surface charge density σ on the conductor is not uniform, in general. The excess charge generates an electric field on the outside of the conductor. That field must be perpendicular to the surface locally, toward the outside where σ is positive and to the inside where σ is negative.

Any tangential component of the electric field would drag mobile charge carriers along the surface, which only happens before equilibrium has been established.

Charged Conductor at Equilibrium (2)

- Now place a point charge q near the charged conductor.
- The electric field produced by q causes a further rearrangement of mobile surface charges until we have again $\vec{E} = 0$ in the interior.
- Locally, the electric field $\vec{E}$ is perpendicular to the surface of the conductor, and its magnitude is proportional to the charge per unit area: $E = \sigma/\epsilon_0$.



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This slide makes two points. We first address the second point.

Consider a clump of conducting material as shown. There is some excess charge Q on it, which will arrange itself spontaneously until the electric field inside the conducting material vanishes and the electric field outside is perpendicular to the surface everywhere.

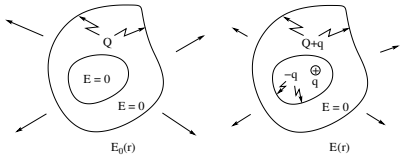
The surface charge density σ, which may vary from point to point, is related to the local electric field E just outside the conductor We already know that the field is directed perpendicular to the surface.

The relation between σ and E uses a tiny Gaussian can placed as shown. The only flux through the can comes from the exterior flat surface and has the value EA, where E is the local field and A the area of the flat surface. The charge inside the can is σA, where σ is the local surface charge density. Gauss's law thus predicts that $E = \sigma/\epsilon_0$. This last relation is generally true for surfaces of conductors at equilibrium.

What happens if we place a point charge q next to the charged conductor as shown? Initially, the field of the point charge will penetrate the conductor and set the mobile charges, i.e. the conduction electrons, in motion. They only settle down when the electric field inside the conductor vanishes again. The excess charge Q on the conductor will now be differently distributed across the surface. However, the field just outside the conductor will again be perpendicular to the surface and the relation $E = \sigma/\epsilon_0$ between field and surface charge density will hold again.

Charged Conductor at Equilibrium (3)

- Consider a conductor with a cavity and excess charge Q.
- Gauss's law implies that there is no net charge on the surface of the cavity.
- The external field is $\vec{E}_0(\vec{r})$. There is no field in the cavity.
- Now place a point charge q inside the cavity.
- Gauss's law implies that there is a charge $-q$ on the surface of the cavity.
- Charge conservation implies that there is a charge $Q+q$ on the outer surface of the conductor.
- The external field changes to $\vec{E}(\vec{r})$. There is a nonzero electric field field inside the cavity.



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What if a conductor has more than one surface such as is the case if the conductor has cavities? Cavities are surrounded by inside surfaces. We shall reason that all excess charge goes to the outside surface if the cavities are empty. There is no charge on inside surfaces then.

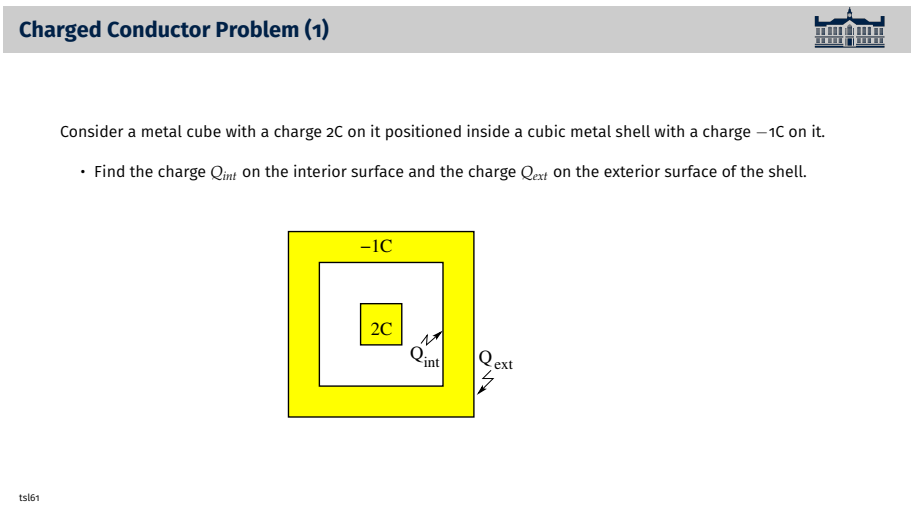
We have not yet developed all the tools to make the argument water tight, but we can show that the net charge on any inside surface is zero. We surround any of the cavities by a Gaussian surface that is embedded in the conducting material, where there is no electric field. Hence the electric flux through that surface vanishes, which implies that the net charge inside is zero. The only conceivable place for charge inside would have been on the cavity wall.

Our reasoning thus far does not rule out regions of positive and of negative surface charge that add up to zero net charge on the cavity surface. We will be able to complete the argument at the end of the next chapter (a few lectures down the line) and show that the surface charge density is zero across the surface of all empty cavities, which has the consequence that there is no electric field in such cavities.

The situation changes when we place a charge into the cavity, for example, a charged particle with positive charge q as shown on the slide. This induces a surface charge $-q$ on the cavity wall as required by Gauss's law. We use the same Gaussian surface embedded in the conducting material and surrounding the cavity to make this point. We reason from no field to no flux to no net charge inside as before. Here no net charge means $q + (-q) = 0$.

Another effect is that the charge on the outer surface of the conductor changes as well, namely from Q to $Q+q$. Placing the charged particle into the cavity must not change the total charge on the conductor. It was Q and remains Q, now split between $Q + q$ on the outer surface and $-q$ on the cavity surface.

Charged Conductor Problem (1)



The next lecture will feature applications of Gauss's law to conductors at equilibrium in various configurations. Here is a sample of what's coming.

The slide shows (in cross section) a metal cube surrounded by a metal cubic shell. Both conductors have been charged up. The cube carries a charge $Q_c = +2\text{C}$ and the shell a charge $Q_s = -1\text{C}$.

From what we have learned thus far, we know that the charge Q_c is located on the surface of the cube. It is not evenly distributed across the surface but none of that charge is in the interior of the cube.

The shell has two surfaces. The question is how the charge Q_s is distributed between the the interior and exterior surfaces.

The trick is to apply Gauss's law by employing a Gaussian surface such that the inside only contains one unknown charge. That means, in this instance, that we imagine a Gaussian surface in the shape of a cube such that it is embedded in the shell.

The electric flux through that Gaussian cube is zero because there is no electric field inside the conducting material. Hence the net charge inside the Gaussian cube must be zero as well, says Gauss's law.

Inside the Gaussian cube we have the known charge $Q_c = +2\text{C}$ on the cube and the unknown charge Q_{int} on the inside surface of the shell. They have to add up to zero, which implies that $Q_{int} = -Q_c = -2\text{C}$.

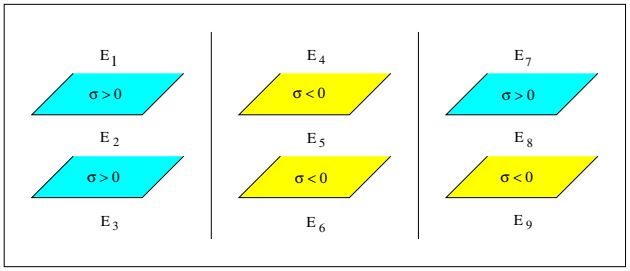
Given the total charge $Q_s = -1\text{C}$ on the shell and the fraction $Q_{int} = -2\text{C}$ of it on its inside surface, we conclude that the remainder is on the outside surface, the only other place where it can be: $Q_{ext} = +1\text{C}$.

Electric Field Near Charged Infinite Sheets

Consider three pairs of parallel, infinite, uniformly charged sheets.

The charge per unit area is equal in magnitude on all sheets.

Find the direction ($\uparrow$, $\downarrow$, none) of the electric field at the nine locations indicated.



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This is the quiz for lecture 6.

We return to pairs of large, uniformly charged sheets positioned parallel to each other as shown in three configurations from left to right on the slide. Positively and negatively charged sheets are color coded.

Both sheets in each configuration generate an electric field in vertical direction, either up or down, away from the sheet if σ is positive and toward the sheet if σ is negative.

Find the direction of the resultant field at the field points marked $E_1, \ldots E_9$.