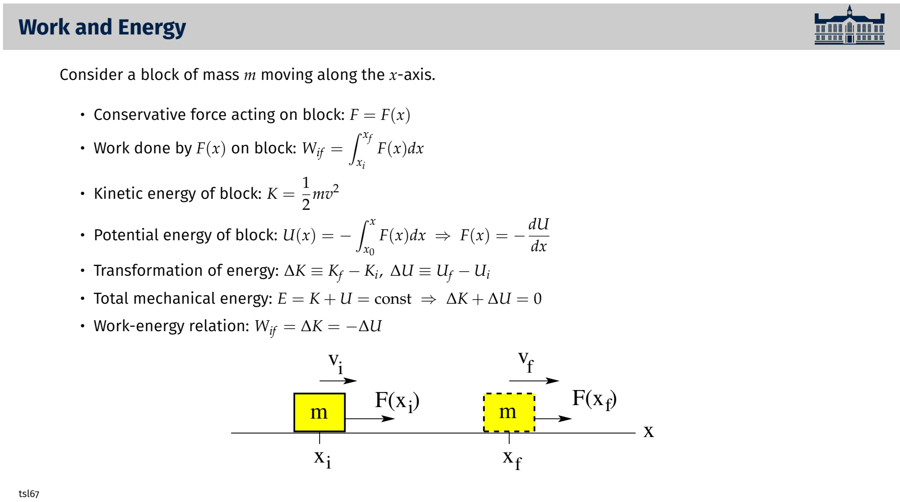
PHY204 Lecture 8 [rln8]



In this lecture we introduce the concept of electric potential. It is of utmost importance that we understand what it means and how it relates to other concepts such as electric field and electric potential energy.

We begin with a brief review of work and energy in the context of a simple mechanical system: the motion of a block under the influence of a force.

The familiar quantities of force, work, kinetic and potential energy, and some relations between them are introduced on the slide.

When the force only depends on position x (not on velocity v), it is guaranteed to be a conservative force, meaning that mechanical energy, the sum of kinetic and potential energy, is conserved.

Note that work is a definite integral and potential energy an indefinite integral. Hence W_{if} is a number and $U(x)$ a function. The change in potential energy ΔU is the difference between two definite integrals and related to the work as stated on the slide. Here are some intermediate steps:

$$\Delta U = U_f - U_i = U(x_f) - U(x_i) = -\int_{x_0}^{x_f} F(x)dx + \int_{x_0}^{x_i} F(x)dx$$
$$= -\int_{x_0}^{x_f} F(x)dx - \int_{x_i}^{x_0} F(x)dx = -\int_{x_i}^{x_f} F(x)dx = -W_{if}.$$

The reference position x_0 can be chosen as is convenient.

Conservative Forces in Mechanics

Conservative forces familiar from mechanics:

- Elastic force: $F(x) = -kx \quad \Rightarrow \; U(x) = -\int_{x_0}^{x}(-kx)dx = \frac{1}{2}kx^2 \qquad (x_0 = 0).$
- Gravitational force (locally): $F(y) = -mg$

$$\Rightarrow \; U(y) = -\int_{y_0}^{y}(-mg)dy = mgy \qquad (y_0 = 0).$$

- Gravitational force (globally): $F(r) = -G\frac{mm_E}{r^2}$

$$\Rightarrow \; U(r) = -\int_{r_0}^{r}\left(-G\frac{mm_E}{r^2}\right)dr = -G\frac{mm_E}{r} \qquad (r_0 = \infty).$$

Potential energy depends on integration constant.

$U = 0$ at reference positions $x_0,\ y_0,\ r_0$.

Force from potential energy: $F(x) = -\frac{d}{dx}U(x), \quad F(y) = -\frac{d}{dy}U(y), \quad F(r) = -\frac{d}{dr}U(r).$

tsl68

Here we focus on the relationship between force F and potential energy U for three familiar situations in mechanics. In all three cases, the force is given as a function of the relevant coordinate. In one case, the force is a constant.

The potential energy is defined as an indefinite integral of the force. It is a function of the same coordinate.

When we perform the integral, we must choose a starting location, which the slide calls the reference position. By construction, the potential energy has the value zero at this position.

We are free to choose the reference position. Some choices are more convenient than others. The impact of different choices is limited to an additive constant, which has no bearing on the physical behavior.

Once the potential energy U as a function of the relevant coordinate is determined as carried out on the slide, the force F as a function of the same coordinate can be recovered via derivative,

$$F(x) = -\frac{dU}{dx}, \quad F(y) = -\frac{dU}{dy}, \quad F(r) = -\frac{dU}{dr},$$

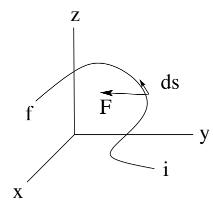
for the three examples, respectively. The force function thus obtained is independent of the choice of reference position used for the calculation of the potential energy.

Work and Potential Energy in 3D Space

Consider a particle acted on by a force $\vec{F}$ as it moves along a specific path in 3D space.

- Force: $\vec{F}(\vec{r}) = F_x(x,y,z)\,\hat{i} + F_y(x,y,z)\,\hat{j} + F_z(x,y,z)\,\hat{k}$
- Displacement: $d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}$
- Potential energy: $U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{s} = -\int_{x_0}^{x} F_x dx - \int_{y_0}^{y} F_y dy - \int_{z_0}^{z} F_z dz$
- Work: $W_{if} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{s} = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz$

Note: The work done by a conservative force is path-independent.



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Things are bit more complicated in three-dimensional space.

The force $\vec{F}(\vec{r})$ is a vector. It has three components, each depending on the three components x, y, z of the position vector $\vec{r}$. The potential energy $U(\vec{r})$ is a scalar function of x, y, z.

We can determine $U(\vec{r})$ by an integration starting from some reference position $\vec{r}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ to an arbitrary point $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

In order to carry out the integration we must choose a path in space (not shown) that connects the points $\vec{r}_0$ and $\vec{r}$. The integral to be performed is spelled out in the third item. The integrand is a dot product of the local force vector (first item) and the displacement vector (second item).

There are infinitely many paths that connect $\vec{r}_0$ and $\vec{r}$. Some choices are more convenient than others. For conservative forces, the result of the integration along any path is the same.

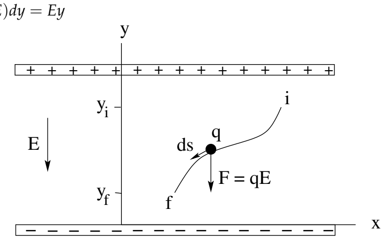
Not all forces that can be written as functions $\vec{F}(\vec{r})$ are conservative. However, a force $\vec{F}$ that only depends on a single spatial coordinate, e.g. x or y or r as used on the previous page, is guaranteed to be conservative.

We will practice in due course how to carry out integrals of dot products along paths in three-dimensional space.

Note that the integral in the fourth item, is a definite integral. It does not yield a function but a number, representing the work done by the (conservative) force that relocates the particle from position $\vec{r}_i$ to position $\vec{r}_f$.

Potential Energy of Charged Particle in Uniform Electric Field

- Electrostatic force: $\vec{F} = -qE\hat{j}$ (conservative)
- Displacement: $d\vec{s} = dx\hat{i} + dy\hat{j}$
- Potential energy: $U = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{s} = -\int_0^y (-qE)dy = qEy$
- Work: $W_{if} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{s} = \int_{y_i}^{y_f} (-qE)dy = -qE(y_f - y_i)$
- Electric potential: $V(y) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{s} = -\int_0^y (-E)dy = Ey$



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A charged particle in an electric field $\vec{E}$ experiences an electric force $\vec{F} = q\vec{E}$. This is a conservative force. Therefore, we can assign a potential energy $U(\vec{r})$ to the particle and calculate it by using the recipe given previously.

We focus on a special situation where the electric field is uniform in a region of space generated by oppositely charged parallel plates as shown. We choose a coordinate system and pick a path (not shown) with the reference point at its origin: $x = y = 0$.

The dot product between the force $\vec{F}$ (first item) and the displacement $d\vec{s}$ (second item) reduces as follows:

$$\vec{F} \cdot d\vec{s} = -qE\,dy.$$

Hence the indefinite integral carried out in the third item yields the potential energy as a function of y. It only depends on that coordinate.

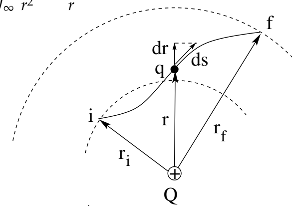
The definite integral along the path shown in the graph yields the work done by the electric force on the charged particles as it relocates from the initial point i to the final point f. The work is independent of the path between the two endpoints and only depends on their y-components.

In the last item we introduce a new quantity, the electric potential. It is an integral constructed similarly to that for potential energy, but only involving the electric field, not the force acting on a charged particle.

The function $V(y)$ is an attribute of space. If we position a particle with charge q at any point with electric potential $V(y)$, that particle has electric potential energy $U(y) = qV(y)$.

Potential Energy of Charged Particle in Coulomb Field

- Electrostatic force: $\vec{F} = \frac{kqQ}{r^2}\hat{r}$ (conservative)
- Displacement: $d\vec{s} = d\vec{r} + d\vec{s}_\perp$, $\quad d\vec{r} = dr\hat{r}$
- Work: $W_{if} = \int_i^f \vec{F}\cdot d\vec{s} = kqQ\int_i^f \frac{\hat{r}\cdot d\vec{s}}{r^2} = kqQ\int_{r_i}^{r_f}\frac{dr}{r^2} = kqQ\left[-\frac{1}{r}\right]_{r_i}^{r_f} = -kqQ\left[\frac{1}{r_f}-\frac{1}{r_i}\right]$
- Potential energy: $U = -\int_{\vec{r}_0}^{\vec{r}}\vec{F}\cdot d\vec{s} = -\int_\infty^r F dr = -kqQ\int_\infty^r \frac{dr}{r^2} = k\frac{qQ}{r}$
- Electric potential: $V(r) = -\int_{\vec{r}_0}^{\vec{r}}\vec{E}\cdot d\vec{s} = -\int_\infty^r E dr = -kQ\int_\infty^r\frac{dr}{r^2} = \frac{kQ}{r}$



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On this page we analyze the same quantities again but for a different electric field, namely the field generated by a fixed point charge Q. A particle with charge q placed into this field experiences a Coulomb force as stated in the first item.

The work done by this force on the particle as it is repositioned from point i to point f is expressed as a definite integral the dot product $\vec{F}\cdot d\vec{s}$ between acting force and displacement.

We can simplify the integrand by splitting the vector of displacement into a a radial part and a part perpendicular to the radial direction (second item). Since the force vector is radial, only the radial part of the displacement contributes to the dot product. The integral is evaluated in the third item.

Since the Coulomb force is a conservative force, we can calculate the potential energy $U(r)$ of the charged particles via an indefinite integral of the same integrand as carried out in the fourth item. It only depends on the radial distance of the charged particle from the source Q of the electric force.

In like manner, we can calculate an electric potential $V(r)$ via an indefinite integral involving the integrand $\vec{E}\cdot d\vec{s}$. The electric potential, which is an attribute of space, then determines the potential energy of a charged particle positioned in that space: $U(r) = qV(r)$.

Note that in the indefinite integrals for $U(r)$ and $V(r)$ we have used a reference point at $r = \infty$, infinitely far a way from the source Q. It is not a necessary choice, but the choice is convenient.

Attributes of Space and of Charged Particles

	planar source	point source	SI unit
electric field	$\vec{E} = -E_y\hat{j}$	$\vec{E} = \frac{kQ}{r^2}\hat{r}$	[N/C]=[V/m]
electric potential	$V = E_y y$	$V = \frac{kQ}{r}$	[V]=[J/C]
electric force	$\vec{F} = q\vec{E} = -qE_y\hat{j}$	$\vec{F} = q\vec{E} = \frac{kQq}{r^2}\hat{r}$	[N]
electric potential energy	$U = qV = qE_y y$	$U = qV = \frac{kQq}{r}$	[J]

Electric field $\vec{E}$ is present at points in space.

Points in space are at electric potential V.

Charged particles experience electric force $\vec{F} = q\vec{E}$.

Charged particles have electric potential energy $U = qV$.

tsl447

This slide aims to drive home key insights gained from the concepts previously introduced in the context of two specific applications.

In classical Newtonian mechanics, all the action is with material objects moving through space while exerting forces on each other. Space itself is understood to be entirely passive, oblivious to the actions of matter.

Advances in physics gradually abandoned this conceptual framework in many respects. The notions of fields and potentials have been early contributions to this shift away from absolute and inert space.

Electric field $\vec{E}(\vec{r})$ and electric potential $V(\vec{r})$ are attributes of space. They are equivalent under certain conditions, as we shall see later, in the sense that one can be derived from the other. One is a vector the other a scalar.

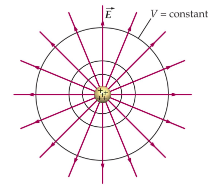
When we place a particle with charge q at position $\vec{r}$ in a region of space where there is an electric field $\vec{E}(\vec{r})$, it experiences an electric force $\vec{F} = q\vec{E}(\hat{r})$.

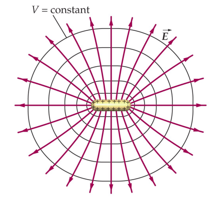
Likewise, when we place a particle with charge q at position $\vec{r}$ in a region of space where there is an electric potential $V(\vec{r})$, it has electric potential energy $U = qV(\hat{r})$.

The table compiles expressions for the four quantities under scrutiny pertaining to the two situations discussed on the previous two pages. Also shown are the SI units of all four quantities.

Equipotential Surfaces and Field Lines

- Definition: $V(\vec{r}) = \text{const}$ on equipotential surface.
- Potential energy $U(\vec{r}) = \text{const}$ for point charge q on equipotential surface.
- The surface of a conductor at equilibrium is an equipotential surface.
- Electric field vectors $\vec{E}(\vec{r})$ (tangents to field lines) are perpendicular to equipotential surface.
- Electrostatic force $\vec{F} = q\vec{E}(\vec{r})$ does zero work on point charge q moving on equipotential surface.
- The electric field $\vec{E}(\vec{r})$ exerts a force on a positive (negative) point charge q in the direction of steepest potential drop (rise).
- When a positive (negative) point charge q moves from a region of high potential to a region of low potential, the electric field does positive (negative) work on it. In the process, the potential energy decreases (increases).





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From a previous lecture, we are already familiar with electric field lines and the information they contain. They tell us what the direction of the field roughly is in a given location and whether the field is strong or weak.

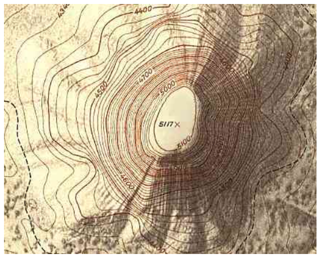
We graphically represent the electric potential, which has no direction, by what are called equipotential surfaces. These are surfaces in three-dimensional space on which the potential has a particular value.

On the slide we show such surfaces in cross section. It can be proven that field lines intersect equipotential surfaces in perpendicular direction. If we move along a field line we move from higher to lower values of potential. Hence, near positive charges we are at high potential and near negative charges at low potential.

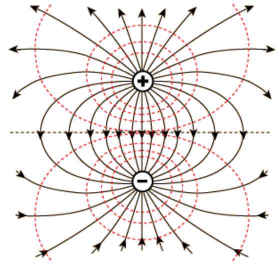
The text on the slide spells out what that means for a charged particle placed into such landscape of electric field and electric potential.

Topographic Maps

Gravitation



Electricity



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It is helpful to visualize the concept of *potential* in the context of a topographic map.

The image on the left shows lines of constant altitude for a piece of landscape with a mountain at its center. Each line is associated with a number in units of feet above sea level. It is straightforward to convert this number into numbers for the gravitational potential $V(y) = gy$, where $g = 9.8\text{m/s}^2$ and y the distance above seal level in units of meters.

In the image on the right, lines of constant electric potential are shown dashed. They represent a topographic map for a piece of landscape in the form of a plane that contain an electric dipole.

The SI unit of gravitational potential is [J/kg] and that of electric potential is [J/C]=[V] (volt).

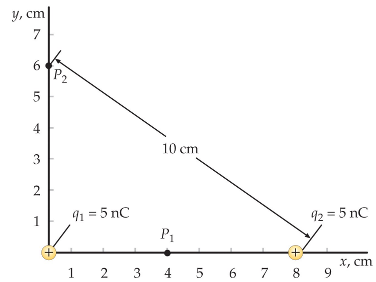
A hiker climbing to higher altitude moves up in gravitational potential. The work exerted in the process is equal to her change in potential energy, which is her mass times the change in gravitational potential: $U = m\Delta V$.

A charged particle that moves against the direction of the local electric field line moves up in electric potential. The work exerted (by the agent that pushes the particle) is equal to the change in potential energy, which is the charge q of the particle times the change in electric potential.

The mass m of the hiker is positive whereas the charge q of the particle can be positive or negative. Therefore, the work required of a hiker to climb to higher gravitational potential is always positive. The work required of an agent to push a proton to higher electric potential is also positive, but it is negative when an electron is being moved in the same direction.

Electric Potential and Potential Energy: Application (2)

- Electric potential at point P_1: $V = \frac{kq_1}{0.04\text{m}} + \frac{kq_2}{0.04\text{m}} = 1125\text{V} + 1125\text{V} = 2250\text{V}.$
- Electric potential at point P_2: $V = \frac{kq_1}{0.06\text{m}} + \frac{kq_2}{0.10\text{m}} = 750\text{V} + 450\text{V} = 1200\text{V}.$



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Here is an application of electric potential to point charges that demonstrates the advantage of working with scalars rather than vectors.

There are two positive point charges positioned on the x-axis as shown. We are being asked to find the electric potential at point P_1 (on the x-axis) and at point P_2 (on the y-axis).

We know that the electric potential generated by a point charge is

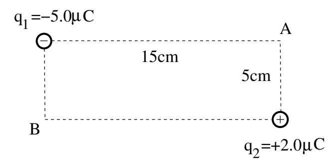
$$V = \frac{kq}{r},$$

where q is the charge of the point source and r is the distance between the source and the field point.

The only thing we have to establish is the distance between each field point and each source. Then we add, for each field point, the contributions to V from the two sources. This simple calculation is carried out on the slide.

Calculating the electric field at the two field point is more cumbersome because we are dealing with vectors, which then have to be added vectorially. We have performed such exercises in an earlier lecture.

Point charges $q_1 = -5.0\mu$C and $q_2 = +2.0\mu$C are positioned at two corners of a rectangle as shown.



(a) Find the electric potential at the corners A and B.
(b) Find the electric field at point B.
(c) How much work is required to move a point charge $q_3 = +3\mu$C from B to A?

tsl75

Parts (a) and (b) of this exercise are what we have done and talked about on the previous page. The electric-field part is simple enough in this case.

$$V_A = k\left[\frac{-5\mu\text{C}}{15\text{cm}} + \frac{2\mu\text{C}}{5\text{cm}}\right] = -30 \times 10^4\text{V} + 36 \times 10^4\text{V} = 6 \times 10^4\text{V}.$$

$$V_B = k\left[\frac{-5\mu\text{C}}{5\text{cm}} + \frac{2\mu\text{C}}{15\text{cm}}\right] = -90 \times 10^4\text{V} + 12 \times 10^4\text{V} = -78 \times 10^4\text{V}.$$

$$\vec{E}_B = -\frac{k(2\mu\text{C})}{(15\text{cm})^2}\,\hat{\mathbf{i}} - \frac{k(-5\mu\text{C})}{(5\text{cm})^2}\,\hat{\mathbf{j}} = -8.0 \times 10^5\text{V/m}\,\hat{\mathbf{i}} + 1.8 \times 10^7\text{V/m}\,\hat{\mathbf{j}}.$$

The minus sign in front of the first two field expressions are due to the fact that the field point is located to the left of one source and below the other source. The minus sign inside the parenthesis of the second expression is due to the fact that the source in this case is negative.

Part (c) is manageable when we recall the relations between work, electric potential energy, and electric potential.

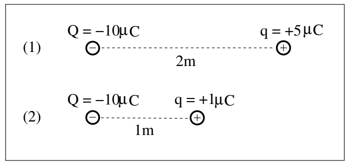
The work done by an external agent to move the charge q_3 from B to A is minus the work done by the electric force on the charge while it is being moved: $W_{ext} = -W_{BA}$.

According to what we have established on the first page of this lecture in a different context, we have: $W_{BA} = -\Delta U$. Finally, we use the relation between the potential energy of the particle and the potential at its position: $U = qV$. Hence we can write,

$$W_{ext} = \Delta U = q_3(V_A - V_B) = (3\mu\text{C})(84 \times 10^4\text{V}) = 2.52\text{J}.$$

Electric Potential and Potential Energy: Application (4)

A positive point charge q is positioned in the electric field of a negative point charge Q.



(a) In which configuration is the charge q positioned in the stronger electric field?
(b) In which configuration does the charge q experience the stronger force?
(c) In which configuration is the charge q positioned at the higher electric potential?
(d) In which configuration does the charge q have the higher potential energy?

tsl76

This is the quiz for lecture 8.

We recall the following relevant quantities:

$$E = \frac{k|Q|}{r^2}, \quad F = \frac{k|Qq|}{r^2}, \quad V = \frac{kQ}{r}, \quad U = \frac{kQq}{r}.$$